MATH3/4/68181: EXTREME VALUES
FIRST SEMESTER
ANSWERS TO IN CLASS TEST

ANSWER TO QUESTION 1 i) If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \rightarrow G(x) \tag{1}$$

as $n \rightarrow \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \quad \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \quad \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0; \\
III &: \quad \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
& \quad \text{for some } \alpha > 0.
\end{aligned}$$

(3 marks)

ii) The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \quad \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: \quad w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: \quad w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

(3 marks)

iii) First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$\lim_{t \rightarrow W(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} = \lim_{t \rightarrow w(F)} \left\{ \frac{1 - \left\{1 - \left[1 - G\left(t + xh(t)\right)\right]^2\right\}^3}{1 - \left\{1 - \left[1 - G\left(t\right)\right]^2\right\}^3} \right\}^4$$

$$
\begin{aligned}
&= \lim_{t \to w(F)} \left\{ \frac{1 - \left\{1 - 3\left[1 - G\left(t + xh(t)\right)\right]^2\right\}}{1 - \left\{1 - 3\left[1 - G\left(t\right)\right]^2\right\}} \right\}^4 \\
&= \lim_{t \to w(F)} \left\{ \frac{\left[1 - G\left(t + xh(t)\right)\right]^2}{\left[1 - G\left(t\right)\right]^2} \right\}^4 \\
&= \lim_{t \to w(F)} \left\{ \frac{1 - G\left(t + xh(t)\right)}{1 - G\left(t\right)} \right\}^8 \\
&= e^{-8x}
\end{aligned}
$$

for every $x \in (-\infty, \infty)$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-8x)\right]
$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(tx\right)}{1 - F(t)} &= \lim_{t \to \infty} \left\{ \frac{1 - \left\{1 - \left[1 - G\left(tx\right)\right]^2\right\}^3}{1 - \left\{1 - \left[1 - G\left(t\right)\right]^2\right\}^3} \right\}^4 \\
&= \lim_{t \to \infty} \left\{ \frac{1 - \left\{1 - 3\left[1 - G\left(tx\right)\right]^2\right\}}{1 - \left\{1 - 3\left[1 - G\left(t\right)\right]^2\right\}} \right\}^4 \\
&= \lim_{t \to \infty} \left\{ \frac{\left[1 - G\left(tx\right)\right]^2}{\left[1 - G\left(t\right)\right]^2} \right\}^4 \\
&= \lim_{t \to \infty} \left\{ \frac{1 - G\left(tx\right)}{1 - G\left(t\right)} \right\}^8 \\
&= x^{8\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left(-x^{-8\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t \to 0} \frac{1 - G\left(w(G) - tx\right)}{1 - G\left(w(G) - t\right)} = x^{\beta}
$$

for every $x > 0$. But But

$$
\begin{aligned}
\lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \downarrow 0} \left\{ \frac{1 - \left\{ 1 - \left[1 - G(w(F) - tx)\right]^2 \right\}^3}{1 - \left\{ 1 - \left[1 - G(w(F) - t)\right]^2 \right\}^3} \right\}^4 \\
&= \lim_{t \downarrow 0} \left\{ \frac{1 - \left\{ 1 - 3\left[1 - G(w(F) - tx)\right]^2 \right\}}{1 - \left\{ 1 - 3\left[1 - G(w(F) - t)\right]^2 \right\}} \right\}^4 \\
&= \lim_{t \downarrow 0} \left\{ \frac{\left[1 - G(w(F) - tx)\right]^2}{\left[1 - G(w(F) - t)\right]^2} \right\}^4 \\
&= \lim_{t \to \infty} \left\{ \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \right\}^8 \\
&= x^{8\beta}
\end{aligned}
$$

for every $x < 0$, assuming $w(F) = w(G)$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \le x \right) = \exp\left(-(-x)^{8\beta} \right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

(4 marks)

ANSWER TO QUESTION 2 i) Note that $w(F) = \infty$. Then

$$
\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - 1 + (tx)^{-2}}{1 - 1 + (t)^{-2}} = \lim_{t \uparrow \infty} \left(\frac{tx}{t} \right)^{-2} = x^{-2}.
$$

. So, the cdf $F(x) = 1 - x^{-2}$ belongs to the Fréchet domain of attraction.

(2 marks)

ii) Note that $w(F) = \infty$. Then

$$
\begin{aligned}
\lim_{t \uparrow \infty} \frac{1 - F(t + xg(t))}{1 - F(t)} &= \lim_{t \uparrow \infty} \frac{1 - 1 + \exp(-t - xg(t))}{1 - 1 + \exp(-t)} \\
&= \lim_{t \uparrow \infty} \frac{\exp(-t - xg(t))}{\exp(-t)} \\
&= \lim_{t \uparrow \infty} \exp(-xg(t)) \\
&= \exp(-x)
\end{aligned}
$$

if $g(t) = 1$. So, the pdf $f(x) = \exp(-x)$ belongs to the Gumbel domain of attraction.

(2 marks)

iii) Note that $w(F) = 1$ and

$$\frac{\Pr(X=1)}{1-F(1-1)} = \frac{\Pr(X=1)}{1-F(0)} = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1.$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

(2 marks)

iv) Note that $w(F) = \infty$. Then

$$\lim_{t\uparrow\infty} \frac{1-F(tx)}{1-F(t)} = \lim_{t\uparrow\infty} \frac{1-\exp\left(-(tx)^{-2}\right)}{1-\exp\left(-t^{-2}\right)} = \lim_{t\uparrow\infty} \frac{1-1+(tx)^{-2}}{1-1+t^{-2}} = \lim_{t\uparrow\infty} \frac{(tx)^{-2}}{t^{-2}} = x^{-2}$$

. So, the cdf $F(x) = \exp\left(-x^{-2}\right)$ belongs to the Fréchet domain of attraction.

(2 marks)

v) Note that $w(F) = \infty$. Then

$$\begin{aligned}
\lim_{t\uparrow\infty} \frac{1-F(t+xg(t))}{1-F(t)} &= \lim_{t\uparrow\infty} \frac{1-\exp\{-\exp(-t-xg(t))\}}{1-\exp\{-\exp(-t)\}} \\
&= \lim_{t\uparrow\infty} \frac{1-1+\exp(-t-xg(t))}{1-1+\exp(-t)} \\
&= \lim_{t\uparrow\infty} \frac{\exp(-t-xg(t))}{\exp(-t)} \\
&= \lim_{t\uparrow\infty} \exp(-t-xg(t)) \\
&= \exp(-x)
\end{aligned}$$

if $g(t) = 1$. So, the cdf $F(x) = \exp\{-\exp(-x)\}$ belongs to the Gumbel domain of attraction.

(2 marks)

ANSWER TO QUESTION 3 i) The cdf of Y is

$$\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= 1-\Pr(Y > y) \\
&= 1-\Pr\left(\min\left(X_1,\ldots,X_m\right) > y\right) \\
&= 1-\Pr\left(X_1 > y,\ldots,X_m > y\right) \\
&= 1-\Pr\left(X_1 > y\right)\cdots\Pr\left(X_m > y\right) \\
&= 1-\left(\frac{b-y}{b-a}\right)\cdots\left(\frac{b-y}{b-a}\right) \\
&= 1-\left(\frac{b-y}{b-a}\right)^m
\end{aligned}$$

for $y > 0$.

(2 marks)

ii) The corresponding pdf is

$$f_Y(y) = \frac{m}{b-a}\left(\frac{b-y}{b-a}\right)^{m-1}.$$

for $y > 0$.

(2 marks)

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iii) The nth moment of Y can be calculated as

$$\begin{aligned}
E\left(Y^n\right) &= \int_a^b y^n \frac{m}{b-a}\left(\frac{b-y}{b-a}\right)^{m-1} dy \\
&= m(b-a)^{-m}\int_a^b \left[b-(b-y)\right]^n (b-y)^{m-1} dy \\
&= m(b-a)^{-m}\int_a^b \sum_{k=0}^{n}\binom{n}{k} b^{n-k}(-1)^k (b-y)^{m+k-1} dy \\
&= m(b-a)^{-m}\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(-1)^k \int_a^b (b-y)^{k+m-1} dy \\
&= m(b-a)^{-m}\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(-1)^k \left[-\frac{(b-y)^{k+m}}{k+m}\right]_a^b \\
&= m(b-a)^{-m}\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(-1)^k \frac{(b-a)^{k+m}}{k+m} \\
&= m\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(-1)^k \frac{(b-a)^{k}}{k+m}.
\end{aligned}$$

So,

$$E\left(Y\right) = m\sum_{k=0}^{1} b^{1-k}(-1)^k \frac{(b-a)^k}{k+m}$$

and

$$Var\left(Y\right) = m\sum_{k=0}^{2}\binom{2}{k} b^{2-k}(-1)^k \frac{(b-a)^k}{k+m} - E^2(Y).$$

(2 marks)

iv) Setting

$$1-\left(\frac{b-y}{b-a}\right)^{m} = p$$

gives

$$\mathrm{VaR}_p(Y) = b - (b-a)(1-p)^{1/m}.$$

(2 marks)

v) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(Y) &= \frac{1}{p}\int_0^p \left(b - (b-a)(1-v)^{1/m}\right) dv \\
&= \frac{1}{p}\left[bv + (b-a)\frac{(1-v)^{1+1/m}}{1+1/m}\right]_0^p \\
&= \frac{1}{p}\left[bp + (b-a)\frac{(1-p)^{1+1/m}-1}{1+1/m}\right].
\end{aligned}$$

(2 marks)