MATH3/4/68181: EXTREME VALUES
FIRST SEMESTER
ANSWERS TO IN CLASS TEST

ANSWER TO QUESTION 1 If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \rightarrow G(x) \tag{1}$$

as $n \rightarrow \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
III &: \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0.
\end{aligned}$$

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{aligned}
I &: \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function say $h(t)$ such that

$$\lim_{t \rightarrow w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = e^{-x}$$

for every $x \in (-\infty, \infty)$. But

$$\begin{aligned}
\lim_{t \rightarrow w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} &= \lim_{t \rightarrow w(G)} \frac{1 - \left\{1 - \left[1 - G\left(t + xh(t)\right)^{\theta}\right]^4\right\}^{\alpha}}{1 - \left\{1 - \left[1 - G\left(t\right)^{\theta}\right]^4\right\}^{\alpha}} \\
&= \lim_{t \rightarrow w(G)} \frac{1 - \left\{1 - \alpha\left[1 - G\left(t + xh(t)\right)^{\theta}\right]^4\right\}}{1 - \left\{1 - \alpha\left[1 - G\left(t\right)^{\theta}\right]^4\right\}}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t \to w(G)} \frac{\left[1 - G\left(t + xh(t)\right)^{\theta}\right]^4}{\left[1 - G\left(t\right)^{\theta}\right]^4} \\
&= \left[\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)^{\theta}}{1 - G\left(t\right)^{\theta}}\right]^4 \\
&= \left[\lim_{t \to w(G)} \frac{1 - \left\{1 - \left[1 - G\left(t + xh(t)\right)\right]\right\}^{\theta}}{1 - \left\{1 - \left[1 - G\left(t\right)\right]\right\}^{\theta}}\right]^4 \\
&= \left[\lim_{t \to w(G)} \frac{1 - \left\{1 - \theta\left[1 - G\left(t + xh(t)\right)\right]\right\}}{1 - \left\{1 - \theta\left[1 - G\left(t\right)\right]\right\}}\right]^4 \\
&= \left[\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G\left(t\right)}\right]^4 \\
&= e^{-4x}
\end{aligned}
$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \exp\left[-\exp(-4x)\right]$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$\lim_{t \to \infty} \frac{1 - G\left(tx\right)}{1 - G(t)} = x^{-\beta}$$

for every $x > 0$. But

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1 - F\left(tx\right)}{1 - F(t)} &= \lim_{t \to \infty} \frac{1 - \left\{1 - \left[1 - G\left(tx\right)^{\theta}\right]^4\right\}^{\alpha}}{1 - \left\{1 - \left[1 - G\left(t\right)^{\theta}\right]^4\right\}^{\alpha}} \\
&= \lim_{t \to \infty} \frac{1 - \left\{1 - \alpha\left[1 - G\left(tx\right)^{\theta}\right]^4\right\}}{1 - \left\{1 - \alpha\left[1 - G\left(t\right)^{\theta}\right]^4\right\}} \\
&= \lim_{t \to \infty} \frac{\left[1 - G\left(tx\right)^{\theta}\right]^4}{\left[1 - G\left(t\right)^{\theta}\right]^4} \\
&= \left[\lim_{t \to \infty} \frac{1 - G\left(tx\right)^{\theta}}{1 - G\left(t\right)^{\theta}}\right]^4 \\
&= \left[\lim_{t \to \infty} \frac{1 - \left\{1 - \left[1 - G\left(tx\right)\right]\right\}^{\theta}}{1 - \left\{1 - \left[1 - G\left(t\right)\right]\right\}^{\theta}}\right]^4 \\
&= \left[\lim_{t \to \infty} \frac{1 - \left\{1 - \theta\left[1 - G\left(tx\right)\right]\right\}}{1 - \left\{1 - \theta\left[1 - G\left(t\right)\right]\right\}}\right]^4
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\lim_{t\to\infty} \frac{1-G(tx)}{1-G(t)}\right]^4 \\
&= x^{-4\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-x^{-4\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\beta > 0$ such that

$$
\lim_{t\to 0} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)} = x^{\beta}
$$

for every $x > 0$. But

$$
\begin{aligned}
\lim_{t\to 0} \frac{1-F(w(F)-tx)}{1-F(w(F)-t)} &= \lim_{t\to 0} \frac{1-\left\{1-\left[1-G(w(G)-tx)^{\theta}\right]^4\right\}^{\alpha}}{1-\left\{1-\left[1-G(w(G)-t)^{\theta}\right]^4\right\}^{\alpha}} \\
&= \lim_{t\to 0} \frac{1-\left\{1-\alpha\left[1-G(w(G)-tx)^{\theta}\right]^4\right\}}{1-\left\{1-\alpha\left[1-G(w(G)-t)^{\theta}\right]^4\right\}} \\
&= \lim_{t\to 0} \frac{\left[1-G(w(G)-tx)^{\theta}\right]^4}{\left[1-G(w(G)-t)^{\theta}\right]^4} \\
&= \left[\lim_{t\to 0} \frac{1-G(w(G)-tx)^{\theta}}{1-G(w(G)-t)^{\theta}}\right]^4 \\
&= \left[\lim_{t\to 0} \frac{1-\{1-[1-G(w(G)-tx)]\}^{\theta}}{1-\{1-[1-G(w(G)-t)]\}^{\theta}}\right]^4 \\
&= \left[\lim_{t\to 0} \frac{1-\{1-\theta[1-G(w(G)-tx)]\}}{1-\{1-\theta[1-G(w(G)-t)]\}}\right]^4 \\
&= \left[\lim_{t\to\infty} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)}\right]^4 \\
&= x^{4\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$
\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \le x\right) = \exp\left(-(-x)^{4\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

ANSWER TO QUESTION 2 i) Note that $w(F)=\infty$. Then

$$\lim_{t\uparrow\infty}\frac{1-F(tx)}{1-F(t)}=\lim_{t\uparrow\infty}\frac{(1+tx)^{-2}}{(1+t)^{-2}}=\lim_{t\uparrow\infty}\left(\frac{1+tx}{1+t}\right)^{-2}=x^{-2}.$$

. So, the cdf $F(x)=1-(1+x)^{-2}$ belongs to the Fréchet domain of attraction.

ii) Note that $w(F)=1<\infty$. Then

$$\lim_{t\to 0}\frac{1-F(1-tx)}{1-F(1-t)}=\lim_{t\to 0}\frac{xf(1-tx)}{f(1-t)}=\lim_{t\to 0}\frac{6xtx(1-tx)}{6t(1-t)}=x^2.$$

So, the pdf $f(x)=6x(1-x)$ belongs to the Weibull domain of attraction.

iii) For the geometric distribution,

$$F(k)=1-(1-p)^k,$$

so

$$\frac{\Pr(X=k)}{1-F(k-1)}=\frac{p(1-p)^{k-1}}{(1-p)^{k-1}}=p.$$

Hence, there can be no sequences $a_n>0$ and b_n such that $(M_n-b_n)/a_n$ has a non-degenerate limiting distribution.

iv) Note that $w(F)=\infty$. Then

$$\begin{aligned}
\lim_{t\to\infty}\frac{1-F(t+xg(t))}{1-F(t)} &= \lim_{t\to\infty}\frac{1-\Phi^2(t+xg(t))}{1-\Phi^2(t)}\\
&= \lim_{t\to\infty}\frac{-2\Phi(t+xg(t))\,\phi(t+xg(t))\left(1+xg^{'}(t)\right)}{-2\Phi(t)\phi(t)}\\
&= \lim_{t\to\infty}\frac{\Phi(t+xg(t))\,\phi(t+xg(t))\left(1+xg^{'}(t)\right)}{\Phi(t)\phi(t)}\\
&= \lim_{t\to\infty}\frac{\phi(t+xg(t))\left(1+xg^{'}(t)\right)}{\phi(t)}\\
&= \lim_{t\to\infty}\frac{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(t+xg(t))^2}{2}\right]\left(1+xg^{'}(t)\right)}{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{t^2}{2}\right]}\\
&= \lim_{t\to\infty}\frac{\exp\left[-\frac{(t+xg(t))^2}{2}\right]\left(1+xg^{'}(t)\right)}{\exp\left[-\frac{t^2}{2}\right]}\\
&= \lim_{t\to\infty}\exp\left[\frac{t^2}{2}-\frac{(t+xg(t))^2}{2}\right]\left(1+xg^{'}(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \exp\left\{-\frac{1}{2}\left[(t+xg(t))^2 - t^2\right]\right\}\left(1+xg^{'}(t)\right) \\
&= \lim_{t\to\infty} \exp\left\{-\frac{1}{2}\left[2txg(t) + x^2g^2(t)\right]\right\}\left(1+xg^{'}(t)\right) \\
&= \lim_{t\to\infty} \exp\left\{-x-\frac{x^2}{2t^2}\right\}\left(1-\frac{x}{t^2}\right) \\
&= \exp(-x)
\end{aligned}
$$

if $g(t) = 1/t$. So, F belongs to the Gumbel domain of attraction.

v) Note that $w(F) = \infty$. Then

$$
\lim_{t\uparrow\infty} \frac{1-F(tx)}{1-F(t)} = \lim_{t\uparrow\infty} \frac{1-\exp\left(-(tx)^{-1}\right)}{1-\exp\left(-t^{-1}\right)} = \lim_{t\uparrow\infty} \frac{1-\left(1-(tx)^{-1}\right)}{1-\left(1-t^{-1}\right)} = \lim_{t\uparrow\infty} \frac{(tx)^{-1}}{t^{-1}} = x^{-1}.
$$

. So, the cdf $F(x) = \exp\left(-x^{-1}\right)$ belongs to the Fréchet domain of attraction.

ANSWER TO QUESTION 3 i) The cdf of Y is

$$
\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr\left(\max\left(X_1,\ldots,X_\alpha\right) \leq y\right) \\
&= \Pr\left(X_1 \leq y,\ldots,X_\alpha \leq y\right) \\
&= \Pr\left(X_1 \leq y\right)\cdots\Pr\left(X_\alpha \leq y\right) \\
&= \left[1-\left(\frac{K}{y}\right)^a\right]\cdots\left[1-\left(\frac{K}{y}\right)^a\right] \\
&= \left[1-\left(\frac{K}{y}\right)^a\right]^\alpha.
\end{aligned}
$$

(2 marks)

ii) The corresponding pdf is

$$
f_Y(y) = a\alpha K^a y^{-a-1}\left[1-\left(\frac{K}{y}\right)^a\right]^{\alpha-1}
$$

for $y \geq K$.

(2 marks)

iii) The nth moment of Y can be calculated as

$$
\begin{aligned}
E\left(Y^n\right) &= a\alpha K^a \int_K^\infty y^{n-a-1}\left[1-\left(\frac{K}{y}\right)^a\right]^{\alpha-1} dy \\
&= a\alpha K^a \int_K^\infty K^{n-a-1}(1-x)^{-\frac{n-a-1}{a}} x^{\alpha-1}\frac{K}{a}(1-x)^{-\frac{1}{a}-1}dx \\
&= K^n \int_0^1 x^{\alpha-1}(1-x)^{-n/a}dx \\
&= K^n B\left(\alpha, 1-\frac{n}{a}\right)
\end{aligned}
$$

provided that $n < a$, where we have set $x = 1 - (K/y)^a$. So,

$$E\left(Y\right) = KB\left(\alpha, 1 - \frac{1}{a}\right)$$

and

$$Var\left(Y\right) = K^2 B\left(\alpha, 1 - \frac{2}{a}\right) - K^2 B^2\left(\alpha, 1 - \frac{1}{a}\right).$$

(2 marks)

iv) Setting

$$\left[1 - \left(\frac{K}{y}\right)^a\right]^\alpha = p$$

gives

$$\mathrm{VaR}_p(Y) = K\left(1 - p^{1/\alpha}\right)^{-1/a}.$$

(2 marks)

v) The expected shortfall is

$$\begin{aligned}
\mathrm{ES}_p(Y) &= \frac{K}{p}\int_0^p \left(1 - v^{1/\alpha}\right)^{-1/a} dv \\
&= \frac{K}{p}\int_0^p \sum_{k=0}^{\infty} \binom{-1/a}{k} (-1)^k v^{k/\alpha} dv \\
&= \frac{K}{p}\int_0^p \sum_{k=0}^{\infty} \binom{-1/a}{k} (-1)^k \int_0^p v^{k/\alpha} dv \\
&= \frac{K}{p}\sum_{k=0}^{\infty} \binom{-1/a}{k} (-1)^k \left[\frac{v^{1+k/\alpha}}{1+k/\alpha}\right]_0^p \\
&= K\sum_{k=0}^{\infty} \binom{-1/a}{k} (-1)^k \frac{p^{k/\alpha}}{1+k/\alpha}.
\end{aligned}$$

(2 marks)