MATH68181: EXTREME VALUES
FIRST SEMESTER
ANSWERS TO IN CLASS TEST

ANSWER TO QUESTION 1 If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n\left(a_n x + b_n\right) \to G(x) \tag{1}$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax+b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{array}{rcl} I & : & \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\ II & : & \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-x^{-\alpha}\right\} & \text{if } x \geq 0 \end{cases} \\ & & \text{for some } \alpha > 0; \\ III & : & \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ & & \text{for some } \alpha > 0. \end{array}$$

The necessary and sufficient conditions for the three extreme value distributions are:

$$\begin{array}{rcl} I & : & \exists \gamma(t) > 0 \text{ s.t. } \lim\limits_{t \uparrow w(F)} \dfrac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\ II & : & w(F) = \infty \text{ and } \lim\limits_{t \uparrow \infty} \dfrac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\ III & : & w(F) < \infty \text{ and } \lim\limits_{t \downarrow 0} \dfrac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0. \end{array}$$

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then, there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t \to w(G)} \frac{1 - G\left(t + xh(t)\right)}{1 - G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But, using L'Hopital's rule, we note that

$$\begin{aligned} & \lim_{t \to w(F)} \frac{1 - F\left(t + xh(t)\right)}{1 - F(t)} \\ = & \lim_{t \to w(F)} \frac{\left(1 + xh^{'}(t)\right) f\left(t + xh(t)\right)}{f(t)} \\ = & \lim_{t \to w(G)} \frac{\left(1 + xh^{'}(t)\right) \left\{-\log\left[1 - G\left(t + xh(t)\right)\right]\right\}^2 g\left(t + xh(t)\right)}{\left\{-\log\left[1 - G(t)\right]\right\}^2 g(t)} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t \to w(G)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)}{g(t)} \left\{ \frac{\log\left[1-G\left(t+xh(t)\right)\right]}{\log\left[1-G(t)\right]} \right\}^2 \\
&= \lim_{t \to w(G)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)}{g(t)} \left\{ \frac{1-G\left(t\right)}{1-G\left(t+xh(t)\right)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)}{g(t)} \right\}^2 \\
&= \lim_{t \to w(G)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)}{g(t)} \left\{ \frac{g\left(t\right)}{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)}{g(t)} \right\}^2 \\
&= \lim_{t \to w(G)} \frac{\left(1+xh^{'}(t)\right) g\left(t+xh(t)\right)}{g(t)} \\
&= \lim_{t \to w(G)} \frac{1-G\left(t+xh(t)\right)}{1-G(t)} \\
&= \exp(-x)
\end{aligned}
$$

for every $x \in (-\infty, \infty)$. So, it follows that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left\{-\exp(-x)\right\}
$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then, there must exist a $\beta > 0$, such that

$$
\lim_{t \to \infty} \frac{1-G(tx)}{1-G(t)} = x^{-\beta}
$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1-F(tx)}{1-F(t)} &= \lim_{t \to \infty} \frac{xf(tx)}{f(t)} \\
&= \lim_{t \to \infty} \frac{x\left\{-\log\left[1-G(tx)\right]\right\}^2 g(tx)}{\left\{-\log\left[1-G(t)\right]\right\}^2 g(t)} \\
&= \lim_{t \to \infty} \frac{xg(tx)}{g(t)} \left\{ \frac{\log\left[1-G(tx)\right]}{\log\left[1-G(t)\right]} \right\}^2 \\
&= \lim_{t \to \infty} \frac{xg(tx)}{g(t)} \left\{ \frac{1-G(t)}{1-G(tx)} \frac{xg(tx)}{g(t)} \right\}^2 \\
&= \lim_{t \to \infty} \frac{xg(tx)}{g(t)} \left\{ \frac{g(t)}{xg(tx)} \frac{xg(tx)}{g(t)} \right\}^2 \\
&= \lim_{t \to \infty} \frac{xg(tx)}{g(t)} \\
&= \lim_{t \to \infty} \frac{1-G(tx)}{1-G(t)} \\
&= x^{\beta}
\end{aligned}
$$

for every $x > 0$. So, it follows that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$
\lim_{n \to \infty} \Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} = \exp\left(-x^{-\beta}\right)
$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, there must exist a $\alpha > 0$, such that

$$\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\alpha}$$

for every $x > 0$. But, using L'Hopital's rule, we note that

$$\begin{aligned}
\lim_{t \to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} &= \lim_{t \to 0} \frac{x f(w(F) - tx)}{f(w(F) - t)} \\
&= \lim_{t \to 0} \frac{x \left\{- \log \left[1 - G(w(F) - tx)\right]\right\}^{a-1} g(w(F) - tx)}{\left\{- \log \left[1 - G(w(F) - tx)\right]\right\}^{2} g(w(F) - t)} \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{\log \left[1 - G(w(F) - tx)\right]}{\log \left[1 - G(w(F) - t)\right]} \right\}^2 \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{1 - G(w(F) - t)}{1 - G(w(F) - tx)} \frac{x g(w(F) - tx)}{g(w(F) - t)} \right\}^2 \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \left\{ \frac{g(w(F) - t)}{x g(w(F) - tx)} \frac{x g(w(F) - tx)}{g(w(F) - t)} \right\}^2 \\
&= \lim_{t \to 0} \frac{x g(w(F) - tx)}{g(w(F) - t)} \\
&= \lim_{t \to 0} \frac{1 - G(w(F) - tx)}{1 - G(w(F) - t)} \\
&= x^{\alpha}.
\end{aligned}$$

So, it follows that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n \to \infty} \Pr \left\{ a_n \left(M_n - b_n \right) \leq x \right\} = \exp \left\{ -(-x)^{\alpha} \right\}$$

for some suitable norming constants $a_n > 0$ and b_n.

ANSWER TO QUESTION 2 i) Note that $w(F) = \infty$ and take $\gamma(t) = 1$. Then

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x)}{1 - F(t)} = \lim_{t \uparrow \infty} \frac{1 - \left[1 - \exp(-t - x)\right]^2}{1 - \left[1 - \exp(-t)\right]^2} = \lim_{t \uparrow \infty} \frac{2 \exp(-t - x)}{2 \exp(-t)} = \exp(-x).$$

. So, the exponentiated exponential cdf $F(x) = [1 - \exp(-x)]^2$ belongs to the Gumbel domain of attraction.

ii) Note that $w(F) = \infty$. Then

$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \lim_{t \to \infty} \frac{1 - \dfrac{\left\{1 - \exp\left(-2\left(t + x\gamma(t)\right)\right)\right\}^2}{0.5 + 0.5 \left\{1 - \exp\left(-2\left(t + x\gamma(t)\right)\right)\right\}^2}}{1 - \dfrac{\left\{1 - \exp(-2t)\right\}^2}{0.5 + 0.5 \left\{1 - \exp(-2t)\right\}^2}}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{1-\dfrac{\{1-\exp\left(-2\left(t+x\gamma(t)\right)\right)\}^2}{0.5+0.5}}{1-\dfrac{\{1-\exp\left(-2t\right)\}^2}{0.5+0.5}} \\
&= \lim_{t\to\infty} \frac{1-\{1-\exp\left(-2\left(t+x\gamma(t)\right)\right)\}^2}{1-\{1-\exp\left(-2t\right)\}^2} \\
&= \lim_{t\to\infty} \frac{1-\{1-2\exp\left(-2\left(t+x\gamma(t)\right)\right)\}}{1-\{1-2\exp\left(-2t\right)\}} \\
&= \lim_{t\to\infty} \frac{2\exp\left(-2\left(t+x\gamma(t)\right)\right)}{2\exp\left(-2t\right)} \\
&= \lim_{t\to\infty} \exp\left(-2x\gamma(t)\right) \\
&= \exp(-x)
\end{aligned}
$$

if $\gamma(t) = 0.5$. So, the exponentiated exponential geometric cdf $F(x) = \frac{\{1-\exp(-2x)\}^2}{1-0.5+0.5\{1-\exp(-2x)\}^2}$ belongs to the Gumbel domain of attraction.

iii) Note that $w(F) = \infty$. Then

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1-F\left(t+x\gamma(t)\right)}{1-F(t)} &= \lim_{t\to\infty} \frac{1-\left\{1-\dfrac{(0.5)^2\exp\left(-4\left(t+x\gamma(t)\right)\right)}{\left[1-0.5\exp\left(-2\left(t+x\gamma(t)\right)\right)\right]^2}\right\}}{1-\left\{1-\dfrac{(0.5)^2\exp\left(-4t\right)}{\left[1-0.5\exp\left(-2t\right)\right]^2}\right\}} \\
&= \lim_{t\to\infty} \frac{\dfrac{(0.5)^2\exp\left(-4\left(t+x\gamma(t)\right)\right)}{\left[1-0.5\exp\left(-2\left(t+x\gamma(t)\right)\right)\right]^2}}{\dfrac{(0.5)^2\exp\left(-4t\right)}{\left[1-0.5\exp\left(-2t\right)\right]^2}} \\
&= \lim_{t\to\infty} \frac{\dfrac{\exp\left(-4\left(t+x\gamma(t)\right)\right)}{\left[1-0.5\exp\left(-2\left(t+x\gamma(t)\right)\right)\right]^2}}{\dfrac{\exp\left(-4t\right)}{\left[1-0.5\exp\left(-2t\right)\right]^2}} \\
&= \lim_{t\to\infty} \frac{\dfrac{\exp\left(-4x\gamma(t)\right)}{\left[1-0.5\exp\left(-2\left(t+x\gamma(t)\right)\right)\right]^2}}{\dfrac{1}{\left[1-0.5\exp\left(-2t\right)\right]^2}} \\
&= \lim_{t\to\infty} \frac{\dfrac{\exp\left(-4x\gamma(t)\right)}{\left[1-0\right]^2}}{\dfrac{1}{\left[1-0\right]^2}} \\
&= \lim_{t\to\infty} \exp\left(-4x\gamma(t)\right) \\
&= \exp(-x)
\end{aligned}
$$

if $\gamma(t) = 1/4$. So, the exponential-negative binomial distribution cdf $F(x) = 1 - \frac{(1-0.5)^2\exp(-4x)}{\left[1-0.5\exp(-2x)\right]^2}$ belongs to the Gumbel domain of attraction.

iv) For the degenerate distribution,

$$p(k) = \begin{cases} 1, & \text{if } k = 2, \\ 0, & \text{if } k \neq 2. \end{cases}$$

So,

$$F(k) = \begin{cases} 1, & \text{if } k \geq 2, \\ 0, & \text{if } k < 2, \end{cases}$$

and

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \begin{cases} 1/1, & \text{if } k = 2, \\ 0/1, & \text{if } 2 < k < 3, \\ 0/0, & \text{if } k \geq 3, \\ 0/1, & \text{if } k < 2. \end{cases}$$

Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

v) For the Poisson distribution,

$$\begin{aligned} \frac{\Pr(X = k)}{1 - F(k-1)} &= \frac{2^k/k!}{\sum_{j=k}^{\infty} 2^j/j!} \\ &= \frac{1}{1 + \sum_{j=k+1}^{\infty} k! 2^{j-k}/j!}. \end{aligned}$$

The term in the denominator of the last term can be rewritten as

$$\sum_{j=1}^{\infty} \frac{2^j}{(k+1)(k+2)\cdots(k+j)} \leq \sum_{j=1}^{\infty} \left(\frac{2}{k}\right)^j = \frac{2/k}{1 - 2/k}$$

(when $k > 2$) and the bound tends to 0 as $k \to \infty$ and so it follows that $p_k/(1 - F(k-1)) \to 1$. Hence, there can be no sequences $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$ has a non-degenerate limiting distribution.

ANSWER TO QUESTION 3 i) The cdf of $M = \max(X, Y, Z)$ is

$$\begin{aligned} F_M(m) &= \Pr(X < m, Y < m, Z < m) \\ &= 1 - \Pr(X > m) - \Pr(Y > m) - \Pr(Z > m) + \Pr(X > m, Y > m) \\ &\quad + \Pr(X > m, Z > m) + \Pr(Y > m, Z > m) - \Pr(X > m, Y > m, Z > m) \\ &= 1 - \left[1 + \frac{m}{2}\right]^{-2} - \left[1 + \frac{m}{2}\right]^{-2} - \left[1 + \frac{m}{2}\right]^{-2} + [1 + m]^{-2} \\ &\quad + [1 + m]^{-2} + [1 + m]^{-2} - \left[1 + \frac{3m}{2}\right]^{-2}. \end{aligned}$$

ii) Differentiating $F_M(m)$ with respect to m gives the pdf of M as

$$\begin{aligned} f_M(m) &= \left[1 + \frac{m}{2}\right]^{-3} + \left[1 + \frac{m}{2}\right]^{-3} + \left[1 + \frac{m}{2}\right]^{-3} - 2\left[1 + m\right]^{-3} \\ &\quad -2\left[1 + m\right]^{-3} - 2\left[1 + m\right]^{-3} \\ &\quad +3\left[1 + \frac{3m}{2}\right]^{-3}. \end{aligned}$$

iii) The nth moment of M can be calculated as

$$
\begin{aligned}
E\left(M^n\right) &= \int_0^\infty m^n \left[1+\frac{m}{2}\right]^{-3} dm + \int_0^\infty m^n \left[1+\frac{m}{2}\right]^{-3} dm \\
&\quad + \int_0^\infty m^n \left[1+\frac{m}{2}\right]^{-3} dm - 2\int_0^\infty m^n \left[1+m\right]^{-3} dm \\
&\quad -2\int_0^\infty m^n \left[1+m\right]^{-3} dm \\
&\quad -2\int_0^\infty m^n \left[1+m\right]^{-3} dm \\
&\quad +3\int_0^\infty m^n \left[1+\frac{3m}{2}\right]^{-3} dm \\
&= 2^{n+1}\int_0^1 x^{1-n}(1-x)^n dx + 2^{n+1}\int_0^1 x^{1-n}(1-x)^n dx \\
&\quad +22^{n+1}\int_0^1 x^{1-n}(1-x)^n dx - \int_0^1 x^{1-n}(1-x)^n dx \\
&\quad -2\int_0^1 x^{1-n}(1-x)^n dx \\
&\quad -2\int_0^1 x^{1-n}(1-x)^n dx \\
&\quad +3\left(\frac{3}{2}\right)^{-n-1}\int_0^1 x^{1-n}(1-x)^n dx \\
&= 22^n B(2-n,n+1) + 22^n B(2-n,n+1) \\
&\quad +22^n B(2-n,n+1) - 2B(2-n,n+1) \\
&\quad -2B(2-n,n+1) \\
&\quad -2B(2-n,n+1) \\
&\quad +2\left(\frac{3}{2}\right)^{-n} B(2-n,n+1).
\end{aligned}
$$

iv) The cdf of $L=\min(X,Y,Z)$ is

$$
\begin{aligned}
F_L(l) &= 1-\Pr(L>l) \\
&= 1-\Pr(X>l,Y>l,Z>l) \\
&= 1-\left[1+\frac{3l}{2}\right]^{-2}.
\end{aligned}
$$

v) Differentiating $F_L(l)$ with respect to l gives the pdf of L as

$$
f_L(l) = 2\left(\frac{3}{2}\right)\left[1+\frac{3l}{2}\right]^{-3}.
$$

vi) The nth moment of L can be calculated as

$$
\begin{aligned}
E\left(L^n\right) &= 2\left(\frac{3}{2}\right)\int_0^\infty l^n \left[1+\frac{3l}{2}\right]^{-3} dl \\
&= 2\left(\frac{3}{2}\right)^{-n} B(2-n,n+1).
\end{aligned}
$$