THE EXPONENTIATED FRÉCHET DISTRIBUTION

1. INTRODUCTION

Gupta *et al.* (1998) introduced the exponentiated exponential (EE) distribution as a generalization of the standard exponential distribution. In particular, the EE distribution is defined by the cumulative distribution function (cdf)

$$F(x) \quad = \quad \{1 - \exp(-\lambda x)\}^{\alpha} \tag{1.1}$$

(for $x > 0$, $\lambda > 0$ and $\alpha > 0$), which is simply the α-th power of the cdf of the standard exponential distribution. The mathematical properties of this EE distribution have been studied in detail by Gupta and Kundu (2001). The aim of this note is to introduce a distribution which generalizes the standard Fréchet distribution in the same way (1.1) generalizes the standard exponential distribution, and to study its mathematical properties. We know that the cdf of the standard Fréchet distribution is:

$$F(x) \quad = \quad \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}$$

for $x > 0$, $\sigma > 0$ and $\lambda > 0$. We define the new distribution by the cdf:

$$F(x) \quad = \quad 1 - \left[1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]^{\alpha} \tag{1.2}$$

for $\alpha > 0$. We refer to (1.2) as the exponentiated Fréchet (EF) distribution. The corresponding pdf is:

$$f(x) \quad = \quad \alpha\lambda\sigma^{\lambda}\left[1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]^{\alpha-1} x^{-(1+\lambda)} \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}. \tag{1.3}$$

The standard Fréchet distribution is the particular case of (1.3) for $\alpha = 1$. Using the series representation

$$(1+z)^{a} \quad = \quad \sum_{j=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} \frac{z^{j}}{j!},$$

(1.3) can be expressed in the mixture form

$$f(x) \quad = \quad \Gamma(\alpha+1)\lambda\sigma^{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!\Gamma(\alpha-k)} x^{-(1+\lambda)} \exp\left\{-(k+1)\left(\frac{\sigma}{x}\right)^{\lambda}\right\}.$$

Like the EE distribution, (1.2) shares an attractive physical interpretation. Suppose that the lifetimes of n-components in a series system are independently and identically distributed according to (1.2). Then it follows that the lifetime of the system also has the EF distribution. An additional motivation comes from the multitude of applications of the Fréchet distribution (which is also known as the extreme value distribution of type II). A recent book by Kotz and Nadarajah (2000), which describes this distribution, lists over fifty applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds and track race records (to mention just a few).

In the rest of this note, we provide a comprehensive description of the mathematical properties of (1.3). We examine the shape of (1.3) and its associated hazard rate function. We derive formulas for the nth moment and the asymptotic distribution of the extreme order statistics. We also consider estimation issues.

2. SHAPE

The first derivative of $\log f(x)$ for the EF distribution is:

$$\frac{d \log f(x)}{dx} = \frac{\lambda \sigma^{\lambda}}{x^{1+\lambda}} \left[1 + \frac{1-\alpha}{\exp\left\{ \left(\frac{\sigma}{x} \right)^{\lambda} \right\} - 1} \right] - \frac{1+\lambda}{x}.$$

Standard calculations based on this derivative show that $f(x)$ exhibits a single mode at $x = x_0$ with $f(0) = f(\infty) = 0$, where x_0 is the solution of $d \log f(x)/dx = 0$. Furthermore, $x_0 > [\alpha \lambda \sigma^{\lambda}/\{1 + \alpha\lambda\}]^{1/\lambda}$ if $0 < \alpha \leq 1$ and $x_0 < \sigma (\log \alpha)^{-1/\lambda}$ if $\alpha > 1$. Figure 1 illustrates some of the possible shapes of f for selected values of α and $\sigma = 1$, $\lambda = 1$.

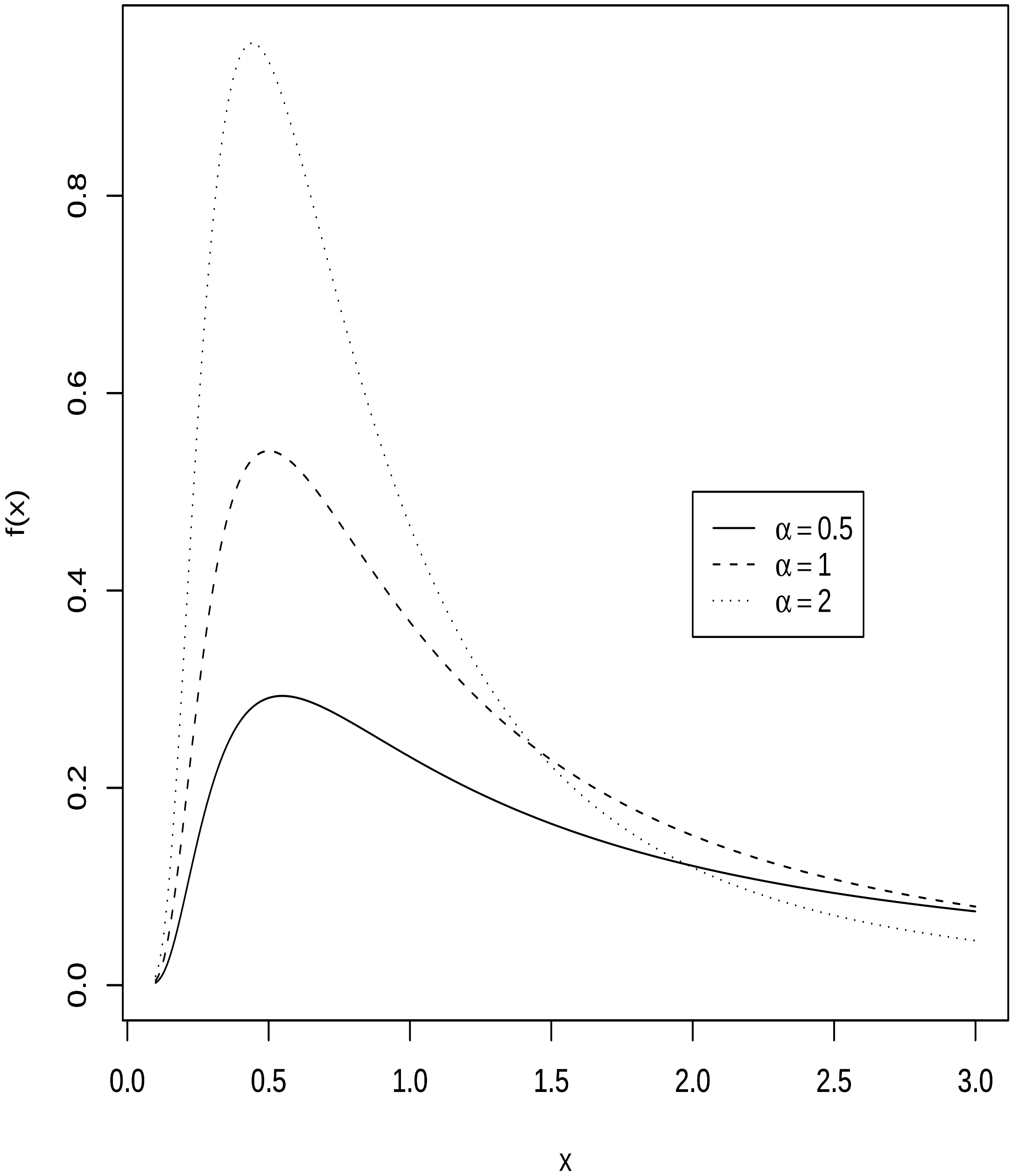

Figure 1. Pdf of the exponentiated Fréchet distribution (1.3) for selected values of α and $\sigma = 1$, $\lambda = 1$.

3. HAZARD RATE FUNCTION

The hazard rate function defined by $h(x) = f(x)/\{1 - F(x)\}$ is an important quantity characterizing life phenomena. For the EF distribution, $h(x)$ takes the form

$$h(x) = \frac{\alpha\lambda\sigma^{\lambda}x^{-(1+\lambda)}\exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}}{1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}}.$$

The first derivative of $\log h(x)$ with respect to x is:

$$\frac{d\log h(x)}{dx} = \frac{\lambda\sigma^{\lambda}x^{-(1+\lambda)}}{1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}} - \frac{1+\lambda}{x}.$$

Standard calculations based on this derivative show that $h(x)$ exhibits a single mode at $x = x_0$ with $h(0) = h(\infty) = 0$, where $y_0 = x_0^{\lambda}$ is the solution of

$$y\left\{1 - \exp\left(-\frac{\sigma^{\lambda}}{y}\right)\right\} = \frac{\lambda\sigma^{\lambda}}{\lambda+1}.$$

Figure 2 illustrates some of the possible shapes of h for selected values of α and $\sigma = 1$, $\lambda = 1$.

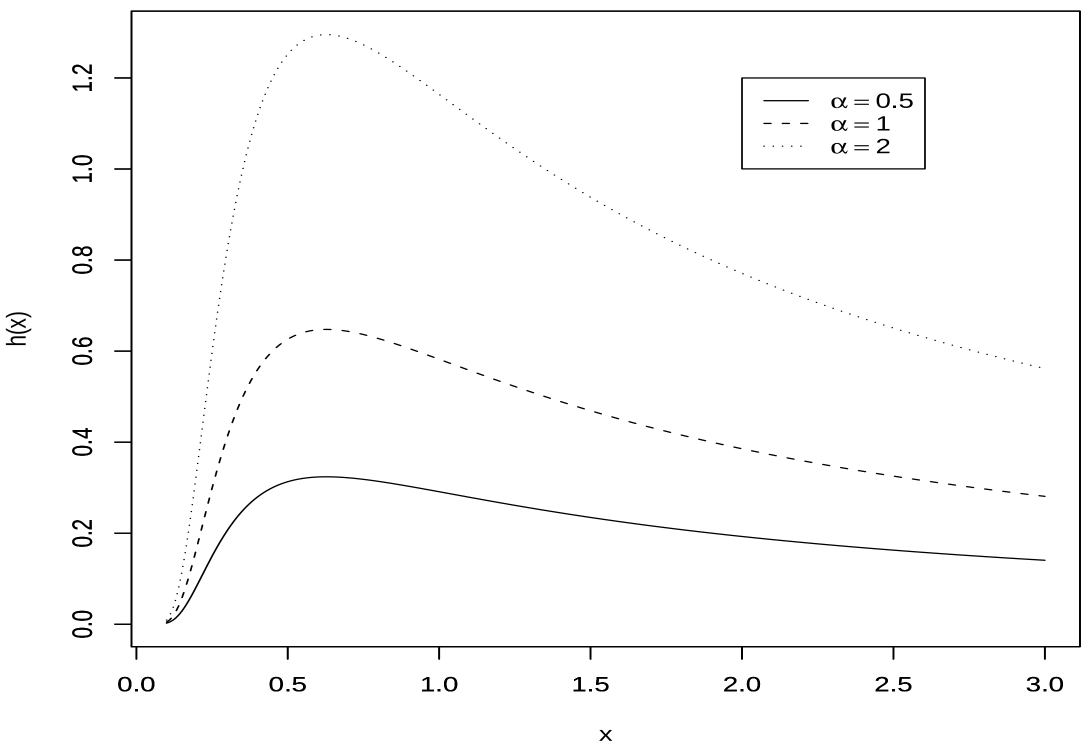

Figure 2. Hazard rate function of the exponentiated Fréchet distribution (1.3) for selected values of α and $\sigma = 1$, $\lambda = 1$.

4. MOMENTS

If X has the pdf (1.3) then by using the well-known relationship

$$\mathrm{E}\left(X^n\right) = \int_0^\infty x^{n-1} \left\{1 - F(x)\right\} dx,$$

the nth moment can be written as

$$\mathrm{E}\left(X^n\right) = \int_0^\infty x^{n-1} \left[1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^\lambda\right\}\right]^\alpha dx. \tag{4.1}$$

On setting $y = (\sigma/x)^\lambda$, (4.1) can be reduced to

$$\mathrm{E}\left(X^n\right) = \frac{\sigma^n}{\lambda} \int_0^\infty y^{-(n/\lambda+1)} \left\{1 - \exp(-y)\right\}^\alpha dy. \tag{4.2}$$

This integral converges if $\alpha > n/\lambda$. However, it is not known how (4.2) can be reduced to a closed-form. The skewness and kurtosis measures can be calculated using (4.2) for all $\alpha > 4/\lambda$. Their variation for $\alpha = 4.1, 4.2, \ldots, 10$ and $\sigma = 1$, $\lambda = 1$ is illustrated in Figure 3.

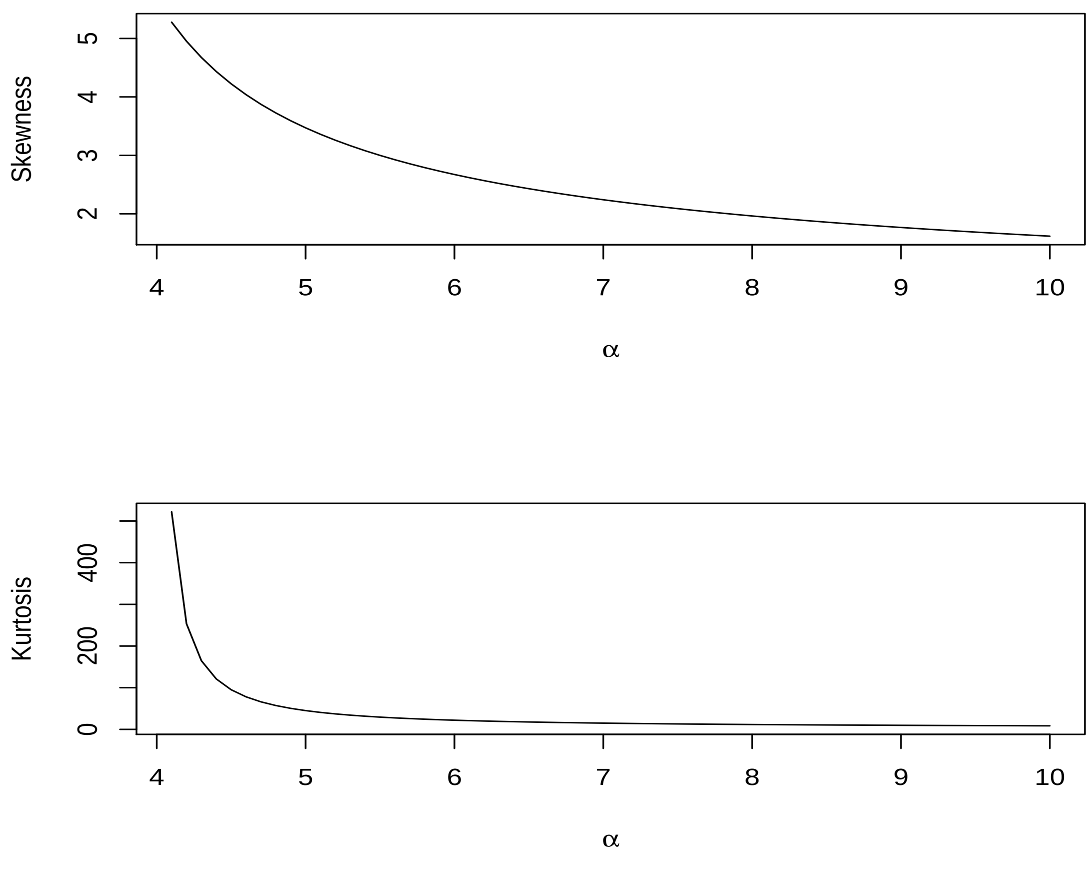

Figure 3. Skewness and kurtosis measures versus $\alpha = 4.1, 4.2, \ldots, 10$ for the exponentiated Fréchet distribution.

It is evident that (1.2) is much more flexible than the standard Fréchet distribution.

5. ASYMPTOTICS

If $X_1, \ldots, X_n$ is a random sample from (1.3) and if $\bar{X} = (X_1 + \cdots + X_n)/n$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}(\bar{X} - \mathrm{E}(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$. Note from (1.3) that

$$1 - F(t) \quad \sim \quad \left(\frac{\sigma}{t} \right)^{\alpha\lambda} \tag{5.1}$$

as $t \to \infty$ and that

$$F(t) \quad \sim \quad \alpha \exp \left\{ - \left(\frac{\sigma}{t} \right)^{\lambda} \right\}$$

as $t \to 0$. Thus, it follows that

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} \quad = \quad x^{-\alpha\lambda}$$

and

$$\lim_{t \to 0} \frac{F\left(t + \sigma^{-\lambda} (x/\lambda) t^{1+\lambda} \right)}{F(t)} \quad = \quad \exp(x).$$

Hence, it follows from Theorem 1.6.2 in Leadbetter *et al.* (1987) that there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$\Pr \left\{ a_n \left(M_n - b_n \right) \leq x \right\} \quad \to \quad \exp\left(-x^{-\alpha\lambda} \right)$$

and

$$\Pr \left\{ c_n \left(m_n - d_n \right) \leq x \right\} \quad \to \quad 1 - \exp \left\{ - \exp(x) \right\}$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter *et al.* (1987), one can see that $b_n = 0$ and that a_n satisfies $1 - F(1/a_n) \sim 1/n$ as $n \to \infty$. Using the fact (5.1), one can see that $a_n = (1/\sigma) n^{-1/(\alpha\lambda)}$ satisfies $1 - F(1/a_n) \sim 1/n$. The constants c_n and d_n can be determined by using the same corollary.

6. ESTIMATION

We consider estimation by the method of maximum likelihood. The log-likelihood for a random sample $x_1, \ldots, x_n$ from (1.3) is:

$$\begin{aligned} \log L(\sigma, \lambda, \alpha) \quad = \quad & n \log \left(\alpha \lambda \sigma^{\lambda} \right) + (\alpha - 1) \sum_{i=1}^{n} \log \left[1 - \exp \left\{ - \left(\frac{\sigma}{x_i} \right)^{\lambda} \right\} \right] \\ & - (1 + \lambda) \sum_{i=1}^{n} \log x_i - \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}. \end{aligned} \tag{6.1}$$

The first order derivatives of (6.1) with respect to the three parameters are:

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} + (\alpha - 1)\lambda\sigma^{\lambda-1} \sum_{i=1}^{n} \frac{\exp\left\{-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right\}}{x_i^{\lambda}\left[1 - \exp\left\{-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right\}\right]} - \lambda\sigma^{\lambda-1}\sum_{i=1}^{n} x_i^{-\lambda},$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n\log\sigma + (\alpha - 1)\sigma^{\lambda} \sum_{i=1}^{n} \frac{\log\left(\frac{\sigma}{x_i}\right)\exp\left\{-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right\}}{x_i^{\lambda}\left[1 - \exp\left\{-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right\}\right]}$$
$$- \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\log\left(\frac{\sigma}{x_i}\right)\left(\frac{\sigma}{x_i}\right)^{\lambda},$$

and

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1 - \exp\left\{-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right\}\right].$$

Setting these expressions to zero and solving them simultaneously yields the maximum likelihood estimates of the three parameters.

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