Extremal Limit Laws for Discrete Random Variables

If $X_1, X_2, \ldots, X_n$ are independent and identically distributed discrete random variables and $M_n = \max(X_1, \ldots, X_n)$ we examine the limiting behavior of $(M_n - b(n))/a(n)$ as $n \to \infty$. It is well known that for discrete distributions such as Poisson and geometric the limiting distribution is not non-degenerate. However, by tuning the parameters of the discrete distribution to vary as $n \to \infty$, it is possible to obtain non-degenerate limits for $(M_n - b(n))/a(n)$. We consider five families of discrete distributions and show how this can be done.

1 Introduction

Let X be a discrete random variable taking the non-negative integers with probability mass function (pmf) $\Pr(X = k) = p_k$ and cumulative distribution function (cdf) F. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (iid) copies of X and let us denote the maximum by $M_n = \max(X_1, \ldots, X_n)$. Consider the limiting distribution of $(M_n - b(n))/a(n)$ as $n \to \infty$ for some real numbers $a(n) > 0$ and $b(n)$. It is known for distributions such as Poisson and geometric the limiting distribution is not non-degenerate. This fact follows by checking the condition in the following result due to Galambos (1987).

Lemma 1 *(Corollary 2.4.1, Galambos, 1987) With the notation set as above, if*

$$\frac{\Pr(X = k)}{1 - F(k-1)} = p_k \Big/ \sum_{j=k}^{\infty} p_j \tag{1}$$

fails to converge to 0 *as* $k \to \infty$*, then there are no sequences* $a(n) > 0$ *and* $b(n)$ *such that* $(M_n - b(n))/a(n)$ *would have a non-degenerate limiting distribution.*

For the Poisson distribution with

$$p_k = \frac{\lambda^k \exp(-\lambda)}{k!}, \qquad k = 0, 1, \ldots,$$

(1) takes the form

$$\frac{\lambda^k / k!}{\sum_{j=k}^{\infty} \lambda^j / j!} = \frac{1}{1 + \sum_{j=k+1}^{\infty} k! \lambda^{j-k} / j!}.$$

The term in the denominator can be rewritten as

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{(k+1)(k+2)\cdots(k+j)} \le \sum_{j=1}^{\infty} \left(\frac{\lambda}{k}\right)^j = \frac{\lambda/k}{1 - \lambda/k}$$

(when $k > \lambda$) and the bound tends to 0 as $k \to \infty$ and so it follows that $p_k/(1-F(k-1)) \to 1$. Hence by the lemma above there can be no non-degenerate limit for $(M_n - b(n))/a(n)$. However, in a recent development Anderson *et al.* (1997) showed that a non-degenerate limit for $(M_n - b(n))/a(n)$ can be obtained if $\lambda = \lambda(n)$ is allowed to vary with respect to n in a suitable manner. More specifically,

Theorem 1 *(Proposition 1, Anderson et al., 1997) Let $X_1, X_2, \ldots, X_n$ be iid Poisson random variables with parameter $\lambda = \lambda(n)$. If $\lambda(n)$ grows with n according to*

$$\log n = o\left(\lambda^{1/3}(n)\right)$$

then there are sequences

$$a(n) = \frac{1}{\sqrt{2\log n}}, \qquad b(n) = \sqrt{2\log n} - \frac{\log\log n + \log 4\pi}{2\sqrt{2\log n}}$$

such that

$$\Pr\left\{M_n \leq \sqrt{\lambda(n)}a(n)x + \sqrt{\lambda(n)}b(n) + \lambda(n)\right\} \rightarrow \exp\left\{-\exp(-x)\right\}$$

as $n \to \infty$.

Actually, the result of this theorem had been proved much earlier by the distinguished Russian mathematician Kolchin (1969), see also Kolchin *et al.* (1978). It is unfortunate that Anderson *et al.* make no mention of this development. In anycase, the aim here is to show how the above development could be emulated for other well known discrete distributions, including the Uniform, Binomial, Geometric, Negative Binomial and the generalized Power Series. The practical motivation for this work comes from the need to model the extremal behavior of processes such as the counts of gamma radiation emission over a fixed period (which can be assumed to follow the Poisson distribution – see Anderson *et al.* (1997)) or the number of failed components at a nuclear site at any given time (which can be assumed to follow the Binomial distribution – see Kvam (1998)) or the counts of new enhancing lesions in untreated multiple sclerosis patients (which can be assumed to follow the Negative Binomial distribution – see Sormani *et al.* (1999)).

Propositions 2, 3 and 4 in Anderson *et al.* (1997) generalize Theorem 1 above under the main assumption that each X_i can be represented as a sum, scaled to zero mean and unit variance, of $k(n)$ iid random variables where $k(n) \to \infty$ in a certain manner. For the four distributions considered, this assumption is valid for the binomial distribution but not for any of the other three (Uniform, Geometric and the Negative Binomial); hence, we also have a theoretical motivation for our work. Although our result for the binomial distribution (Theorem 3) can be derived from the general results of Anderson *et al.* (1997), we prove it directly using the theorem for large deviations for the binomial distribution.

2 Results

In this section we derive non-degenerate limit laws for the Uniform, Binomial, Geometric, Negative Binomial and the generalized Power Series distributions (by letting their parameters vary as $n \to \infty$). We also present plots that provide a numerical assessment of the corresponding rates of convergence. The proofs of all the theorems in this section are presented in Section 3.

The pmf of the uniform distribution is:

$$p_k = \frac{1}{N}, \qquad k = 1, 2, \ldots, N$$

and the corresponding cdf is:

$$F(x) \quad = \quad \frac{[x]}{N}, \qquad x \geq 1, \tag{2}$$

where $[x]$ is the greatest integer less or equal to x. Since

$$\frac{\Pr(X = k)}{1 - F(k-1)} = \frac{1/N}{1 - (k-1)/N} = \frac{1}{N - k + 1} \to 1$$

as $k \to N$, it follows by Lemma 1 that there can be no sequences $a(n) > 0$ and $b(n)$ such that $(M_n - b(n))/a(n)$ has a non-degenerate limiting distribution. The following theorem establishes a non-degenerate limit for $(M_n - b(n))/a(n)$ by letting $N = N(n) \to \infty$ in a certain manner.

Theorem 2 *Let $X_1, X_2, \ldots, X_n$ be iid Uniform random variables with parameter $N = N(n)$. If $N(n)$ grows with n according to*

$$n \quad = \quad o(N(n))$$

then there are sequences

$$a(n) = \alpha \frac{N(n)}{n}, \qquad b(n) = N(n) - \beta \frac{N(n)}{n} \tag{3}$$

(where $\alpha > 0$ and $\beta \geq 0$) such that

$$\Pr\{M_n \leq a(n)x + b(n)\} \quad \to \quad \exp(\alpha x - \beta) \tag{4}$$

as $n \to \infty$.

Figure 1 provides an assessment of the rate of convergence in (4) by comparing the distribution function of $(M_n - b(n))/a(n)$ (the stair-case line) with its limiting distribution (the continuous line), for $n = 10, 100, 1,000$ and $10,000$.

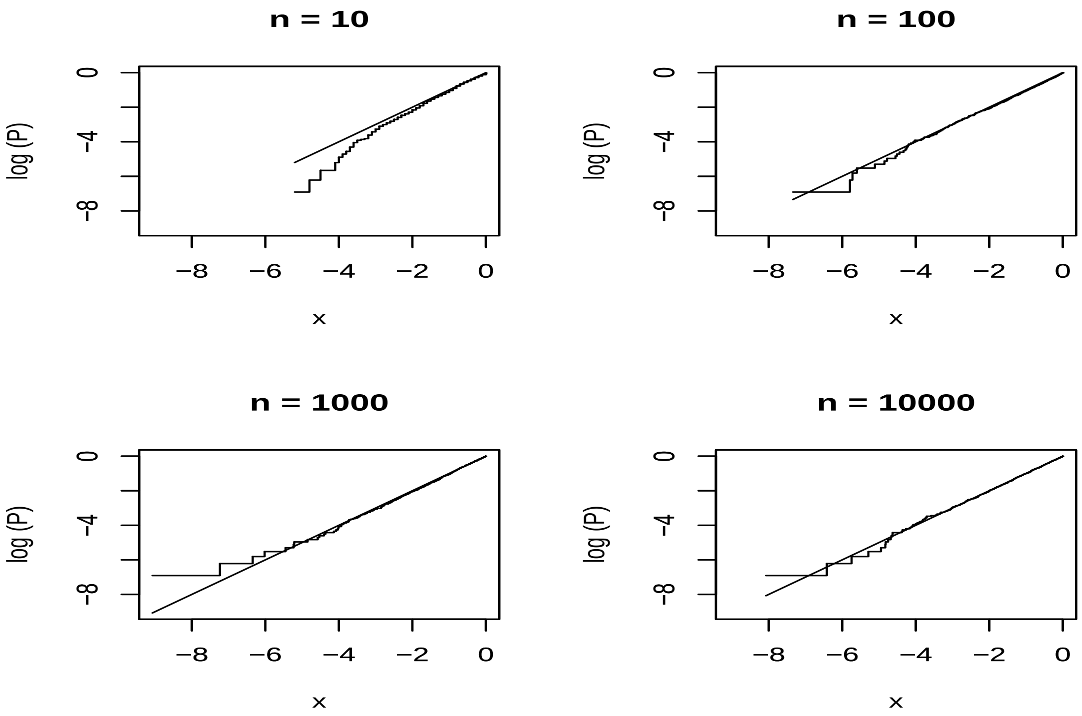

Figure 1: Distribution function of $(M_n - b(n))/a(n)$ for iid Uniform random variables with $N(n) = n^2$, $\alpha = 1$ and $\beta = 0$. Log vertical scale used.

The pmf of the binomial distribution is:

$$p_k = \binom{N}{k} p^k (1-p)^{N-k}, \qquad k = 0, 1, \ldots, N.$$

It is well known (see, for example, equation (62) in Johnson and Kotz (1969)) that

$$\frac{\Pr(X \geq k)}{\binom{N}{k} p^k (1-p)^{N-k}} \leq \frac{(1-p)(k+1)}{(k+1) - (N+1)p}.$$

Using this inequality, we have

$$p_k \Big/ \sum_{j=k}^{\infty} p_j \geq \frac{(k+1) - (N+1)p}{(1-p)(k+1)}$$

and the lower bound approaches 1 as $k \to N$. So by Lemma 1 there can be no sequences $a(n) > 0$ and $b(n)$ such that $(M_n - b(n))/a(n)$ has a non-degenerate limiting distribution. The following theorem establishes a non-degenerate limit for $(M_n - b(n))/a(n)$ by letting $N = N(n) \to \infty$ but keeping the parameter p fixed.

Theorem 3 *Let $X_1, X_2, \ldots, X_n$ be iid Binomial random variables with parameter $N = N(n) \to \infty$ but parameter p fixed. If $N(n)$ grows with n according to*

$$(\log n)^3 = o(N(n))$$

then there are sequences

$$a(n) = \frac{1}{\sqrt{2 \log n}}, \qquad b(n) = \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} \tag{5}$$

such that

$$\Pr\left\{ M_n \leq \sqrt{p(1-p)N(n)}\, a(n) x + pN(n) + \sqrt{p(1-p)N(n)}\, b(n) \right\} \to \exp\{-\exp(-x)\} \tag{6}$$

as $n \to \infty$.

Figure 2 provides an assessment of the rate of convergence in (6) by comparing the distribution function of $(M_n - b(n))/a(n)$ (the stair-case line) with its limiting distribution (the continuous line), for $n = 10, 100, 1,000$ and $10,000$.

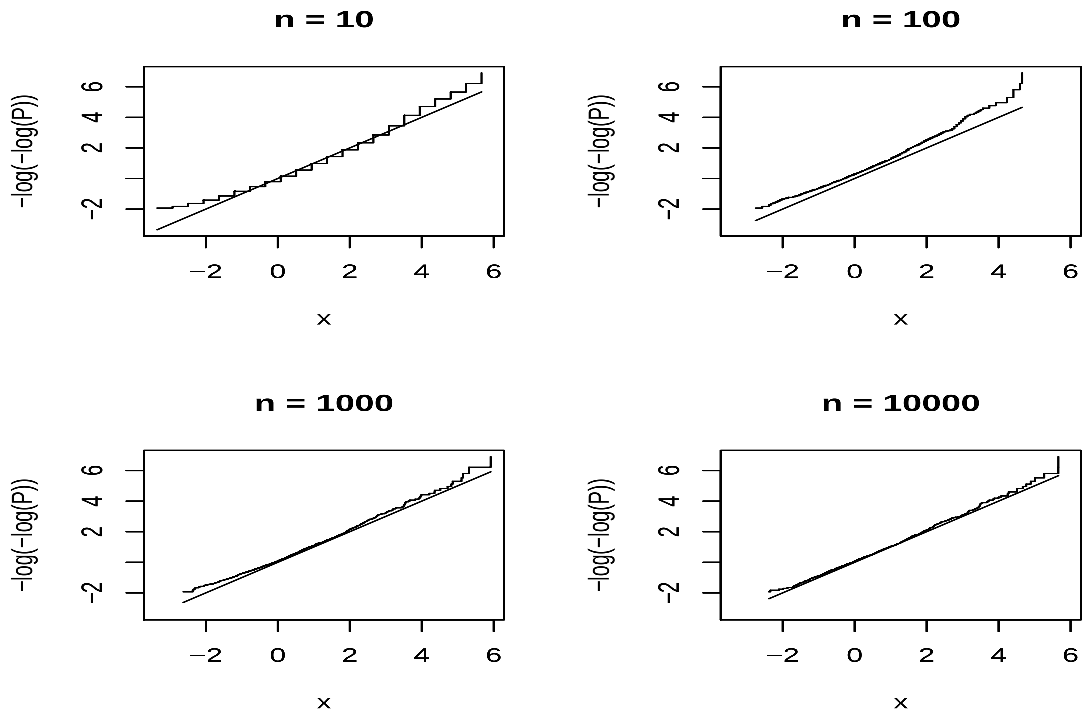

Figure 2: Distribution function of $(M_n - b(n))/a(n)$ for iid Binomial random variables with $N(n) = n^2$ and $p = 1/2$. Double log vertical scale used.

The pmf of the geometric distribution is:

$$p_k \;=\; p(1-p)^{k-1}, \qquad k = 1, 2, \ldots$$

and the corresponding cdf is

$$F(x) \;=\; 1 - (1-p)^{[x]}, \qquad x \geq 1. \tag{7}$$

Since

$$\frac{p_k}{1 - F(k-1)} = \frac{p(1-p)^{k-1}}{(1-p)^{k-1}} = p,$$

it follows by Lemma 1 that there can be no sequences $a(n) > 0$ and $b(n)$ such that $(M_n - b(n))/a(n)$ has a non-degenerate limiting distribution. The following theorem establishes a non-degenerate limit for $(M_n - b(n))/a(n)$ by letting $p = p(n) \to 0$.

Theorem 4 *Let $X_1, X_2, \ldots, X_n$ be iid Geometric random variables with parameter $p = p(n)$. If $p(n) \to 0$ as $n \to \infty$ then there are sequences*

$$a(n) = \frac{\alpha}{p(n)}, \qquad b(n) = \frac{\log(n/\beta)}{p(n)} \tag{8}$$

(where $\alpha > 0$ and $\beta > 0$) such that

$$\Pr\{M_n \leq a(n)x + b(n)\} \;\to\; \exp\{-\beta \exp(-\alpha x)\} \tag{9}$$

as $n \to \infty$.

Figure 3 provides an assessment of the rate of convergence in (9) by comparing the distribution function of $(M_n - b(n))/a(n)$ (the stair-case line) with its limiting distribution (the continuous line), for $n = 10, 100, 1,000$ and $10,000$.

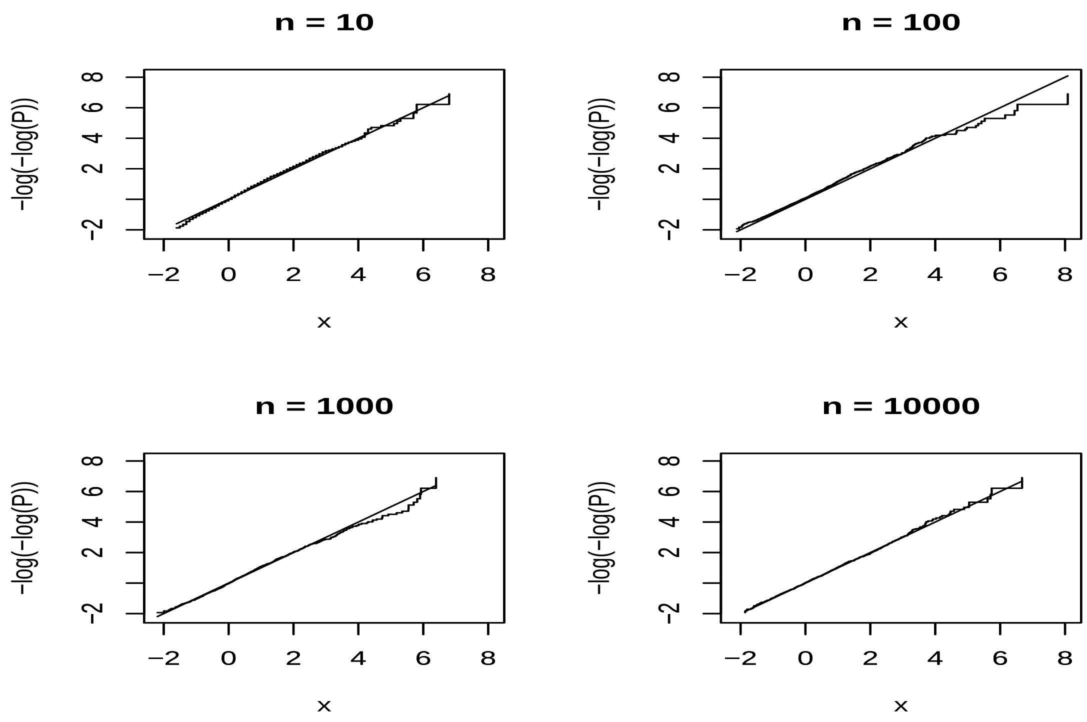

Figure 3: Distribution function of $(M_n - b(n))/a(n)$ for iid Geometric random variables with $p(n) = 1/n$, $\alpha = 1$ and $\beta = 1$. Double log vertical scale used.

The pmf of the negative binomial distribution is:

$$p_k = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \qquad k = r, r+1, \ldots .$$

It is well known that if X is a negative binomial random variable with parameters r and p and if Y is a binomial random variable with parameters N and p then $\Pr(X > N) = \Pr(Y < r)$. Using this result, we have for $x \geq r$

$$\begin{aligned} F(x) &= \Pr\left(X \leq [x]\right) \\ &= 1 - \Pr\left(X > [x]\right) \\ &= 1 - \Pr(Y < r) \\ &= 1 - \sum_{k=0}^{r-1} \binom{[x]}{k} p^k (1-p)^{[x]-k} \\ &= 1 - p^{r-1}(1-p)^{[x]-r+1} \sum_{l=0}^{r-1} \binom{[x]}{r-l-1} \left(\frac{1-p}{p}\right)^l . \end{aligned} \tag{10}$$

It is easily seen that

$$\frac{p_k}{1-F(k-1)} = \frac{\binom{k-1}{r-1} p^r (1-p)^{k-r}}{p^{r-1}(1-p)^{k-r} \sum_{l=0}^{r-1} \binom{k-1}{r-l-1} \left(\frac{1-p}{p}\right)^l} \to p$$

as $k \to \infty$. So by Lemma 1 there can be no sequences $a(n) > 0$ and $b(n)$ such that $(M_n - b(n))/a(n)$ has a non-degenerate limiting distribution. The following theorem establishes a non-degenerate limit for $(M_n - b(n))/a(n)$ by letting $p = p(n) \to 0$ in a certain manner, but keeping r fixed.

Theorem 5 *Let $X_1, X_2, \ldots, X_n$ be iid Negative Binomial random variables with parameter $r \geq 2$ (fixed) and parameter $p = p(n)$. If $p(n) \to 0$ according to*

$$p(n) \quad = \quad o\left(\frac{1}{\log n}\right)$$

then there are sequences

$$a(n) = \frac{\alpha}{p(n)}, \qquad b(n) = \frac{\log n + (r-1)\log\log n - \log(r-1)!}{p(n)} \tag{11}$$

(where $\alpha > 0$) such that

$$\Pr\left\{M_n \leq a(n)x + b(n)\right\} \quad \to \quad \exp\left\{-\exp\left(-\alpha x\right)\right\} \tag{12}$$

as $n \to \infty$.

Figure 4 provides an assessment of the rate of convergence in (12) by comparing the distribution function of $(M_n - b(n))/a(n)$ (the stair-case line) with its limiting distribution (the continuous line), for $n = 10, 100, 1,000$ and $10,000$.

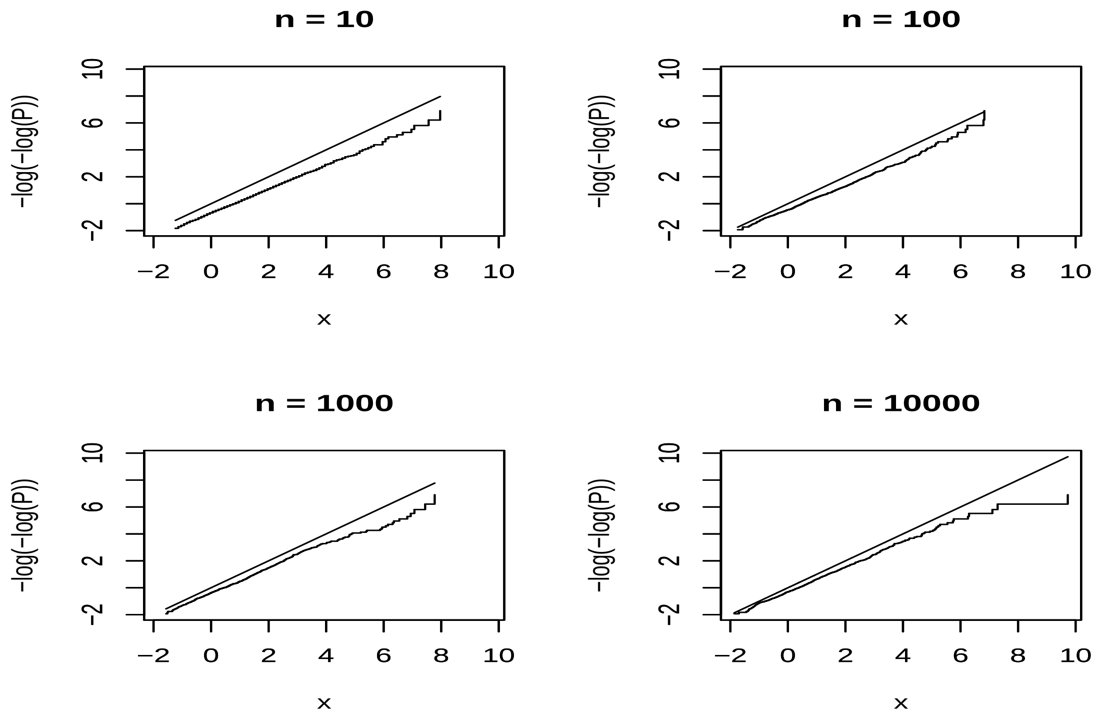

Figure 4: Distribution function of $(M_n - b(n))/a(n)$ for iid Negative Binomial random variables with $r = 2$, $p(n) = 1/n$ and $\alpha = 1$. Double log vertical scale used.

The pmf of the generalized power series distribution is:

$$p_k = \frac{k^\beta (1-p)^k}{C(p)}, \qquad k = 0, 1, \ldots$$

where $0 < p < 1$, $\beta > 0$ and

$$C(p) = \sum_{k=0}^{\infty} k^\beta (1-p)^k,$$

a normalizing constant that depends on p. The corresponding cdf is:

$$\begin{aligned} F(x) &= 1 - \frac{1}{C(p)} \sum_{j=[x]+1}^{\infty} j^\beta (1-p)^j \\ &= 1 - \frac{1}{C(p)} \sum_{j=1}^{\infty} (j + [x])^\beta (1-p)^{j+[x]}. \end{aligned} \tag{13}$$

This family of distributions was introduced by Noack (1950). It contains as special cases the Uniform, Binomial, Geometric and the Negative Binomial distributions. To determine whether or not $(M_n - b(n))/a(n)$ can have a non-degenerate limit, note that we can write

$$\begin{aligned} \frac{p_k}{1 - F(k-1)} &= \frac{k^\beta}{\sum_{j=0}^{\infty} (j+k)^\beta (1-p)^j} \\ &= \frac{1}{\sum_{j=0}^{\infty} (1 + j/k)^\beta (1-p)^j} \\ &= \frac{1}{\sum_{j=0}^{k-1} (1 + j/k)^\beta (1-p)^j + \sum_{j=k}^{\infty} (1 + j/k)^\beta (1-p)^j}. \end{aligned} \tag{14}$$

Since the series

$$\sum_{j=0}^{\infty} (1 + j/k)^\beta (1-p)^j$$

is convergent uniformly on k, we have

$$\sum_{j=k}^{\infty} (1 + j/k)^\beta (1-p)^j \rightarrow 0 \tag{15}$$

as $k \to \infty$. For the first sum in the denominator of (14), since $0 \le j/k < 1$, we can apply the Taylor's expansion to write

$$\sum_{j=0}^{k-1} (1 + j/k)^\beta (1-p)^j = \sum_{j=0}^{k-1} (1-p)^j + O\left(\frac{\beta}{k} \sum_{j=0}^{k-1} j (1-p)^j \right).$$

Now note that

$$\sum_{j=0}^{k-1} (1-p)^j = \frac{1 - (1-p)^k}{p} \rightarrow \frac{1}{p}, \tag{16}$$

and

$$\frac{\beta}{k}\sum_{j=0}^{k-1} j(1-p)^j \to 0 \tag{17}$$

as $k \to \infty$, the latter follows because the series

$$\sum_{j=0}^{\infty} j(1-p)^j$$

is convergent. Combining (15), (16) and (17), we get the limit of (14) as $k \to \infty$ as p. So by Lemma 1 there can be no sequences $a(n) > 0$ and $b(n)$ such that $(M_n - b(n))/a(n)$ has a non-degenerate limiting distribution. The following theorem establishes a non-degenerate limit for $(M_n - b(n))/a(n)$ by letting $p = p(n) \to 0$, but keeping the parameter β fixed.

Theorem 6 *Let $X_1, X_2, \ldots, X_n$ be iid generalized Power Series random variables with parameter β fixed and parameter $p = p(n)$. If $p(n) \to 0$ according to*

$$p(n) \quad = \quad o\left(\frac{1}{\log n}\right)$$

then there are sequences

$$a(n) = \frac{\alpha}{p(n)}, \qquad b(n) = \frac{\log n + \beta \log\log n - \log\Gamma(\beta+1)}{p(n)} \tag{18}$$

(where $\alpha > 0$ and $\beta > 0$) such that

$$\Pr\left\{M_n \le a(n)x + b(n)\right\} \quad \to \quad \exp\left\{-\exp\left(-\alpha x\right)\right\} \tag{19}$$

as $n \to \infty$.

3 Proofs

We need a lemma to help us find the non-degenerate limiting distribution of $(M_n - b(n))/a(n)$.

Lemma 2 *(Theorem 1.5.1, Leadbetter et al., 1987) With the notation set as above, if $a(n) > 0$ and $b(n)$ are sequences of real numbers such that*

$$n\left\{1 - F\left(a(n)x + b(n)\right)\right\} \quad \to \quad -\log H(x), \qquad n \to \infty \tag{20}$$

then

$$\Pr\left\{M_n \le a(n)x + b(n)\right\} \quad \to \quad H(x), \qquad n \to \infty.$$

Proof of Theorem 2. For the cdf (2), using $x-1 \leq [x] \leq x$, we can write

$$n\left(1-\frac{a(n)x+b(n)}{N(n)}\right) \leq n\left\{1-F\left(a(n)x+b(n)\right)\right\} \leq n\left(1-\frac{a(n)x+b(n)-1}{N(n)}\right). \tag{21}$$

If we choose $a(n)>0$ and $b(n)$ as given in (3) then we have

$$n\left(1-\frac{a(n)x+b(n)}{N(n)}\right) = n\left(1-\frac{b(n)}{N(n)}\right) - \frac{na(n)}{N(n)}x \to \beta - \alpha x$$

as $n \to \infty$. The right hand side of (21)

$$n\left(1-\frac{a(n)x+b(n)-1}{N(n)}\right) = -n\frac{a(n)}{N(n)}x + n\left(1-\frac{b(n)}{N(n)}\right) + \frac{n}{N(n)},$$

approaches the same limit since $n = o(N(n))$. So for the sequences $a(n)$ and $b(n)$ given by (3) we have

$$n\left\{1-F\left(a(n)x+b(n)\right)\right\} \quad \to \quad \beta - \alpha x$$

as $n \to \infty$. Hence (4) follows by Lemma 2.

Proof of Theorem 3. We follow the method used by Anderson *et al.* (1997) for Poisson distribution. By the theorem for large deviations for Binomial random variables (see the theorem on page 178 of Feller (1966, volume 1)) we have

$$1-F\left(pN(n)+\sqrt{p(1-p)N(n)}x\right) \quad \sim \quad 1-\Phi(x), \qquad n \to \infty \tag{22}$$

if $N(n) \to \infty$ and $x = x(n) \to \infty$ in such way that

$$x^3/\sqrt{p(1-p)N(n)} \to 0, \qquad \text{i.e.} \qquad x = o\left(N^{1/6}(n)\right)$$

as $n \to \infty$. On the other hand for the standard normal distribution function $\Phi(x)$ we have (see Leadbetter *et al.* (1983), Theorem 1.5.3)

$$n\left\{1-\Phi\left(a(n)x+b(n)\right)\right\} \quad \to \quad \exp(-x), \qquad n \to \infty \tag{23}$$

for all x, where the sequences $a(n)$ and $b(n)$ are given by (5). Since $(\log n)^3 = o(N(n))$ we get by combining (22) and (23) that

$$n\left\{1-F\left(N(n)p+\sqrt{p(1-p)N(n)}x\right)\right\} \sim n\left\{1-\Phi\left(a(n)x+b(n)\right)\right\} \to \exp(-x)$$

as $n \to \infty$. Hence by Lemma 2 the sequences $a(n)$ and $b(n)$ given by (5) will yield (6).

Proof of Theorem 4. For the cdf (7), using $x-1 \leq [x] \leq x$, we can write

$$n\left\{1-p(n)\right\}^{a(n)x+b(n)} \leq n\left\{1-F\left(a(n)x+b(n)\right)\right\} \leq n\left\{1-p(n)\right\}^{a(n)x+b(n)-1}. \tag{24}$$

For the $a(n)$ and $b(n)$ given by (8)

$$\left\{1-p(n)\right\}^{a(n)x} \quad \to \quad \exp(-\alpha x)$$

and

$$n\{1-p(n)\}^{b(n)} \quad \to \quad \beta$$

as $n \to \infty$. So the lower bound in (24) approaches $\beta \exp(-\alpha x)$ as $n \to \infty$. The upper bound in (24) has the same limit since

$$n\{1-p(n)\}^{a(n)x+b(n)-1} \quad = \quad n\{1-p(n)\}^{a(n)x}\{1-p(n)\}^{b(n)-1}$$

and $p(n) \to 0$. Hence by Lemma 2, we have (9).

Proof of Theorem 5. For the cdf (10), using $x-1 \leq [x] \leq x$, we can write

$$\begin{aligned}
& n\{1-F(a(n)x+b(n))\} \\
\leq\ & n\frac{\{p(n)\}^{r-1}}{(r-1)!}\{1-p(n)\}^{a(n)x+b(n)-r} \\
& \quad \times \Bigg[\{a(n)x+b(n)\}\{a(n)x+b(n)-1\}\cdots\{a(n)x+b(n)-r+2\} \\
& \quad +(r-1)\frac{1-p(n)}{p(n)} \\
& \qquad \times \{a(n)x+b(n)\}\{a(n)x+b(n)-1\}\cdots\{a(n)x+b(n)-r+3\} \\
& \quad \vdots \\
& \quad +(r-1)(r-2)\cdots(r-k)\left\{\frac{1-p(n)}{p(n)}\right\}^{k} \\
& \qquad \times \{a(n)x+b(n)\}\{a(n)x+b(n)-1\}\cdots\{a(n)x+b(n)-r+k+2\} \\
& \quad \vdots \\
& \quad +(r-1)(r-2)\cdots 1\left\{\frac{1-p(n)}{p(n)}\right\}^{r-1} \Bigg]
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
& n\{1-F(a(n)x+b(n))\} \\
\geq\ & n\frac{\{p(n)\}^{r-1}}{(r-1)!}\{1-p(n)\}^{a(n)x+b(n)-r+1} \\
& \quad \times \Bigg[\{a(n)x+b(n)-1\}\{a(n)x+b(n)-2\}\cdots\{a(n)x+b(n)-r+1\} \\
& \quad +(r-1)\frac{1-p(n)}{p(n)} \\
& \qquad \times \{a(n)x+b(n)-1\}\{a(n)x+b(n)-2\}\cdots\{a(n)x+b(n)-r+2\} \\
& \quad \vdots \\
& \quad +(r-1)(r-2)\cdots(r-k)\left\{\frac{1-p(n)}{p(n)}\right\}^{k} \\
& \qquad \times \{a(n)x+b(n)-1\}\{a(n)x+b(n)-2\}\cdots\{a(n)x+b(n)-r+k+1\} \\
& \quad \vdots \\
& \quad +(r-1)(r-2)\cdots 1\left\{\frac{1-p(n)}{p(n)}\right\}^{r-1} \Bigg].
\end{aligned} \tag{26}$$

The upper bound (25) can be rewritten as:

$$
\begin{aligned}
& \frac{1}{(r-1)!} \{1-p(n)\}^{a(n)x}\, n \{p(n)b(n)\}^{r-1} \{1-p(n)\}^{b(n)} \{1-p(n)\}^{-r} \\
& \times \Bigg[\left(\frac{a(n)x}{b(n)}+1\right) \left(\frac{a(n)x}{b(n)}+1-\frac{1}{b(n)}\right) \cdots \left(\frac{a(n)x}{b(n)}+1-\frac{r-2}{b(n)}\right) \\
& \quad +(r-1)\frac{1-p(n)}{p(n)b(n)} \\
& \quad \times \left(\frac{a(n)x}{b(n)}+1\right) \left(\frac{a(n)x}{b(n)}+1-\frac{1}{b(n)}\right) \cdots \left(\frac{a(n)x}{b(n)}+1-\frac{r-3}{b(n)}\right) \\
& \quad \vdots \\
& \quad +(r-1)(r-2)\cdots(r-k) \left\{\frac{1-p(n)}{p(n)b(n)}\right\}^{k} \\
& \quad \times \left(\frac{a(n)x}{b(n)}+1\right) \left(\frac{a(n)x}{b(n)}+1-\frac{1}{b(n)}\right) \cdots \left(\frac{a(n)x}{b(n)}+1-\frac{r-k-2}{b(n)}\right) \\
& \quad \vdots \\
& \quad +(r-1)(r-2)\cdots 1 \left\{\frac{1-p(n)}{p(n)b(n)}\right\}^{r-1} \Bigg]. \qquad (27)
\end{aligned}
$$

For the choice of $a(n)$ and $b(n)$ given by (11),

$$\{1-p(n)\}^{a(n)x} \quad \to \quad \exp\{-\alpha x\}$$

and

$$\frac{1}{(r-1)!} n \{p(n)b(n)\}^{r-1} \{1-p(n)\}^{b(n)} \quad \to \quad 1$$

as $n \to \infty$. Since $p(n) \to 0$, we have $\{1-p(n)\}^{-r} \to 1$. Furthermore, since $b(n) \to \infty$, $p(n)b(n) \to \infty$ and $a(n)/b(n) \to 0$, we have

$$\left(\frac{a(n)x}{b(n)}+1\right) \left(\frac{a(n)x}{b(n)}+1-\frac{1}{b(n)}\right) \cdots \left(\frac{a(n)x}{b(n)}+1-\frac{r-2}{b(n)}\right) \quad \to \quad 1$$

and

$$\left\{\frac{1-p(n)}{p(n)b(n)}\right\}^{k} \left(\frac{a(n)x}{b(n)}+1\right) \left(\frac{a(n)x}{b(n)}+1-\frac{1}{b(n)}\right) \cdots \left(\frac{a(n)x}{b(n)}+1-\frac{r-k-2}{b(n)}\right) \quad \to \quad 0$$

for $k = 1, 2, \ldots, r-1$. Substituting these limiting relations into (27), we get

$$\limsup_{n\to\infty} n \{1 - F(a(n)x + b(n))\} \quad \le \quad \exp\{-\alpha x\}.$$

Manipulating the lower bound (26), we can similarly establish that

$$\liminf_{n\to\infty} n \{1 - F(a(n)x + b(n))\} \quad \ge \quad \exp\{-\alpha x\}.$$

Hence (12) follows by Lemma 2.

Proof of Theorem 6. For the cdf (13), using $x - 1 \le [x] \le x$, we can write

$$
\begin{aligned}
& \frac{n}{C(p(n))} \sum_{k=1}^{\infty} \{k + a(n)x + b(n) - 1\}^{\beta} \{1 - p(n)\}^{k+a(n)x+b(n)} \\
\le \; & n \{1 - F(a(n)x + b(n))\} \qquad (28) \\
\le \; & \frac{n}{C(p(n))} \sum_{k=1}^{\infty} \{k + a(n)x + b(n)\}^{\beta} \{1 - p(n)\}^{k+a(n)x+b(n)-1}.
\end{aligned}
$$

The upper bound of (28) can be rewritten as

$$
\begin{aligned}
& \frac{n}{C(p(n))} \{b(n)\}^{\beta} \{1 - p(n)\}^{a(n)x+b(n)-1} \left\{ \sum_{1 \le k \le [a(n)x+b(n)]} \left(1 + \frac{k + a(n)x}{b(n)}\right)^{\beta} \{1 - p(n)\}^{k} \right. \\
& \left. + \sum_{k > [a(n)x+b(n)]} \left(1 + \frac{k + a(n)x}{b(n)}\right)^{\beta} \{1 - p(n)\}^{k} \right\} = \Pi(n) \{S_1(n) + S_2(n)\}, \qquad (29)
\end{aligned}
$$

say. In the sequel we shall show that

$$
\frac{\Pi(n)}{p(n)} \to \exp(-\alpha x), \qquad (30)
$$

$$
p(n) S_1(n) \to 1 \qquad (31)
$$

and

$$
p(n) S_2(n) \to 0 \qquad (32)
$$

as $n \to \infty$. First, consider $\Pi(n)$. By Theorem 5 in Feller (1966, volume 2, page 423), we have

$$
C(p(n)) = \sum_{k=0}^{\infty} k^{\beta} \{1 - p(n)\}^{k} = \frac{\Gamma(\beta + 1)}{p(n)^{\beta+1}} (1 + o(1))
$$

as $n \to \infty$. Substituting this into the form for $\Pi(n)$, we get

$$
\Pi(n) = \frac{n p(n)}{\Gamma(\beta + 1)} \{p(n) b(n)\}^{\beta} \{1 - p(n)\}^{b(n)} \{1 - p(n)\}^{a(n)x} (1 + o(1)) \qquad (33)
$$

as $n \to \infty$. Under the assumed forms for $a(n)$, $b(n)$ and $p(n)$, it is easily seen that

$$
\{1 - p(n)\}^{a(n)x} \to \exp(-\alpha x) \qquad (34)
$$

and

$$
n \{p(n) b(n)\}^{\beta} \{1 - p(n)\}^{b(n)} \to \Gamma(\beta + 1) \qquad (35)
$$

as $n \to \infty$. Substituting these into (33), we have (30). Next, consider $p(n)S_1(n)$. Since $a(n) = o(b(n))$ note that $0 \le \{k + a(n)x\}/b(n) \le 3$ holds for $k \le a(n)x + b(n)$ for all sufficiently large n, so we can apply the Taylor's formula $(1 + x)^{\beta} = 1 + O(x)$ to the terms of $S_1(n)$. We get

$$
\begin{aligned}
& p(n) S_1(n) \\
= \; & p(n) \sum_{1 \le k \le [a(n)x+b(n)]} \{1 - p(n)\}^{k} + O\left(p(n) \sum_{1 \le k \le [a(n)x+b(n)]} \frac{k + a(n)x}{b(n)} \{1 - p(n)\}^{k} \right) \\
= \; & S_{11}(n) + S_{12}(n),
\end{aligned}
$$

say. Note that we can rewrite

$$S_{11}(n) \;=\; \{1-p(n)\}\left[1-\{1-p(n)\}^{[a(n)x+b(n)]}\right] \tag{36}$$

and

$$\begin{aligned}
S_{12}(n) &\le Kp(n)\sum_{1\le k\le[a(n)x+b(n)]}\frac{k+a(n)x}{b(n)}\{1-p(n)\}^k\\
&\le K\frac{p(n)}{b(n)}\sum_{1\le k\le[a(n)x+b(n)]}k\{1-p(n)\}^k\\
&\quad +K\frac{p(n)a(n)x}{b(n)}\sum_{1\le k\le[a(n)x+b(n)]}\{1-p(n)\}^k\\
&= \frac{K}{p(n)b(n)}\{1-p(n)\}\Big[1-\{[a(n)x+b(n)]+1\}\{1-p(n)\}^{a(n)x+b(n)}\\
&\quad +[a(n)x+b(n)]\{1-p(n)\}^{[a(n)x+b(n)]+1}\Big]\\
&\quad +K\frac{a(n)x}{b(n)}\{1-p(n)\}\left[1-\{1-p(n)\}^{[a(n)x+b(n)]}\right],
\end{aligned} \tag{37}$$

for some $K>0$. Using $x-1\le[x]\le x$ to bound (36) and then applying the limiting relations (34) and (35), we see that $S_{11}(n)\to 1$ as $n\to\infty$. Similarly, the terms in (37) approach 0 as $n\to\infty$. Hence we have established (31). Finally, consider $p(n)S_2(n)$. For all sufficiently large n, we can rewrite

$$\begin{aligned}
p(n)S_2(n) &= p(n)\sum_{k>[a(n)x+b(n)]}\left(1+\frac{k+a(n)x}{b(n)}\right)^\beta\{1-p(n)\}^k\\
&= p(n)\sum_{k=0}^{\infty}\left(1+\frac{k+a(n)x+[a(n)x+b(n)]+1}{b(n)}\right)^\beta\{1-p(n)\}^{k+[a(n)x+b(n)]+1}\\
&\le p(n)\{1-p(n)\}^{a(n)x+b(n)}\sum_{k=0}^{\infty}\left(2+\frac{k+2a(n)x+1}{b(n)}\right)^\beta\{1-p(n)\}^k\\
&\qquad\text{(using } x-1\le[x]\le x\text{)}\\
&\le \frac{Kp(n)}{n}\sum_{k=0}^{\infty}\left(3+\frac{k}{b(n)}\right)^\beta\{1-p(n)\}^k\\
&\qquad\text{(using (34) and (35) and since } a(n)=o(b(n))\text{)}\\
&\le \frac{Kp(n)}{n}\sum_{k=0}^{\infty}\left(3+\frac{k}{b(n)}\right)^m\{1-p(n)\}^k\\
&\qquad\text{(putting } m=[\beta]+1\text{)}\\
&= \frac{Kp(n)}{n}\sum_{k=0}^{\infty}\left[\sum_{l=0}^{m}\binom{m}{l}3^{m-l}\left(\frac{k}{b(n)}\right)^l\{1-p(n)\}^k\right]\\
&= \frac{Kp(n)}{n}\sum_{l=0}^{m}\binom{m}{l}\frac{3^{m-l}}{\{b(n)\}^l}\left[\sum_{k=0}^{\infty}k^l\{1-p(n)\}^k\right]\\
&\le \frac{Kp(n)}{n}\sum_{l=0}^{m}\binom{m}{l}\frac{3^{m-l}}{\{b(n)\}^l}\left[\sum_{k=0}^{\infty}(k+l)(k+l-1)\cdots(k+1)\{1-p(n)\}^k\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{Kp(n)}{n}\sum_{l=0}^{m}\binom{m}{l}\frac{3^{m-l}}{\{b(n)\}^{l}}\left[l!\sum_{k=0}^{\infty}\frac{(l+1+k-1)(l+1+k-2)\cdots(l+1)}{k!}\{1-p(n)\}^{k}\right] \\
&= \frac{K}{n}\sum_{l=0}^{m}\binom{m}{l}\frac{3^{m-l}l!}{\{p(n)b(n)\}^{l}}, \qquad (38)
\end{aligned}
$$

the last step follows from equation (10.4.6) in Hansen (1975). The limit of (38) as $n \to \infty$ is 0 because $b(n)p(n) \to \infty$. Hence we have proved (32). Substituting (30), (31) and (32) into (29), we get

$$
\limsup_{n\to\infty} n\left\{1 - F\left(a(n)x + b(n)\right)\right\} \quad \le \quad \exp\left\{-\alpha x\right\}.
$$

Manipulating the lower bound of (28), we can similarly establish that

$$
\liminf_{n\to\infty} n\left\{1 - F\left(a(n)x + b(n)\right)\right\} \quad \ge \quad \exp\left\{-\alpha x\right\}.
$$

Hence (19) follows by Lemma 2.

References

Anderson, C. W., Coles, S. G. and Hüsler, J. (1997). Maxima of Poisson-like variables and related triangular arrays. *Annals of Applied Probability*, **7**, 953–971.

Feller, W. (1966). An Introduction to Probability Theory and Its Applications, volume 1 (second edition). John Wiley, New York.

Feller, W. (1966). An Introduction to Probability Theory and Its Applications, volume 2 (second edition). John Wiley, New York.

Galambos, J. (1987). The Asymptotic Theory of Extreme Order Statistics. Robert E. Krieger Publishing Company, Florida.

Hansen, E. R. (1975). A Table of Series and Products. Prentice-Hall, New Jersey.

Johnson, N. L. and Kotz, S. (1969). Discrete Distributions. John Wiley, New York.

Kolchin, V. F. (1969). The limiting behavior of extreme terms of a variational series in a polynomial scheme. *Theory of Probability and Its Applications*, **14**, 458–469.

Kolchin, V. F., Sevastcyanov, B. A. and Chistyakov, V. P. (1978). Random Allocations. Translated from Russian. Translation Edited by A. V. Balakrishnan. Scripta Series in Mathematics. V. H. Winston and Sons, Washington, D.C.

Kvam, P. H. (1998). The binomial failure rate mixture model for common cause failure data from the nuclear industry. *Applied Statistics*, **47**, 49–61.

Leadbetter, M. R., Lindgren, G. and Rootzén, H. (1987). Extremes and Related Properties of Random Sequences and Processes. Springer Verlag, New York.

Noack, A. (1950). A class of random variables with discrete distributions. *Annals of Mathematical Statistics*, **21**, 127–132.

Sormani, M. P., Bruzzi, P., Miller, D. H., Gasperini, C., Barkhof, F. and Filippi, L. (1999). Modelling MRI enhancing lesion counts in multiple sclerosis using a negative binomial model: implications for clinical trials. *Journal of the Neurological Sciences*, **163**, 74–80.