A compendium of copulas

1 Introduction

A p-dimensional copula is a function $C : [0,1]^p \rightarrow [0,1]$ that satisfies i) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_p) = 0$ for all $1 \leq i \leq p$; ii) $C(1, \ldots, 1, u, 1, \ldots, 1) = u$ for u in each of the p arguments; iii) for $a_i \leq b_i$, $i = 1, \ldots, p$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_p=1}^{2} (-1)^{i_1+\cdots+i_p} C\left(u_{1,i_1}, \ldots, u_{p,i_p}\right) \geq 0,$$

where $u_{j,1} = a_j$ and $u_{j,2} = b_j$ for $j = 1, \ldots, p$.

A copula can be used to specify a multivariate distribution and every multivariate distribution gives a copula. If F_i, $i = 1, \ldots, p$ are one-dimensional cumulative distribution functions then

$$F(x_1, \ldots, x_p) = C(F_1(x_1), \ldots, F_p(x_p))$$

is a p-dimensional cumulative distribution function. If F is a p-dimensional cumulative distribution function then

$$C(u_1, \ldots, u_p) = F\left(F_1^{-1}(u_1), \ldots, F_p^{-1}(u_p)\right)$$

is a p-dimensional copula, where $0 \leq u_i \leq 1$, $i = 1, \ldots, p$ and $F_i(x) = F(\infty, \ldots, \infty, x, \infty, \ldots, \infty)$, $i = 1, \ldots, p$ with x being the ith argument. If

$$C(u_1, \ldots, u_p) = u_1 \cdots u_p$$

then the distribution is said to exhibit independence. If

$$C(u_1, \ldots, u_p) = \min(u_1, \ldots u_p)$$

then the distribution is said to exhibit complete dependence.

The concept of copulas was introduced by Sklar (1959). Since then many parametric, non-parametric and semi-parametric models have been proposed for copulas, including methods for constructing models for copulas. Most of proposed models have been parametric models. There are fewer non-parametric models and even fewer semi-parametric models.

Applications of copulas are too numerous to list. Some recent applications have been: simulation of multivariate sea storms (Corbella and Stretch, 2013); dependence structure between the stock and foreign exchange markets (Wang *et al.*, 2013); operational risk management (Arbenz, 2013); portfolio optimization in the presence of dependent financial returns with long memory (Boubaker and Sghaier, 2013); risk evaluation of droughts across the Pearl River basin, China (Zhang *et al.*, 2013); probabilistic assessment of flood risks (Ganguli and Reddy, 2013); estimation of distribution algorithms for coverage problem of wireless sensor network (Wang *et al.*, 2012); risk assessment of hydroclimatic variability on groundwater levels in the Manjara basin aquifer in India (Reddy and Ganguli, 2012); models of tourists' time use and expenditure behavior with self-selection (Zhang *et al.*, 2012); modeling wind speed dependence in system reliability assessment using copulas (Xie *et al.*, 2012); dependence between crude oil spot and futures markets (Chang, 2012); stochastic modeling of power demand (Lojowska *et al.*, 2012).

Most applications have been based on parametric models for copulas. There are not many applications based on non-parametric or semi-parametric models. Nevertheless, the parametric models used have been very limited (for example, Archimedean copulas). This is possibly due to the practitioners not being aware of the range of parametric copulas available.

The aim of this notes is to provide an up-to-date and a comprehensive collection of known parametric copulas. We feel that such a review is timely because most models for copulas have been proposed in the last five years or so. We feel also that such a review could serve as an important reference, encourage more of the copulas being applied and encourage further developments of copulas.

2 The collection

Here, we provide a list of known parametric copulas, including their characterizations. The copulas are grouped into eighty four sections and number over one hundred. The list is by no means complete, but we believe we have covered most if not all of the important parametric copulas.

2.1 Gaussian copula

Let $\Phi(\cdot)$ denote the distribution function of a standard normal random variable and let $\Phi^{-1}(\cdot)$ denote its inverse. The Gaussian copula with variance-covariance matrix $\mathbf{\Sigma}$ is defined by

$$C\left(u_1, \ldots, u_p\right) = \Phi_{\mathbf{\Sigma}}\left(\Phi^{-1}\left(u_1\right), \ldots, \Phi^{-1}\left(u_p\right)\right),$$

where $\Phi_{\mathbf{\Sigma}}$ denotes distribution function of a p-variate normal random vector with zero means and variance-covariance matrix $\mathbf{\Sigma}$. A perturbed version of Gaussian copula is presented in Fouque and Zhou (2008).

2.2 t copula

Let $t_\nu(\cdot)$ denote the distribution function of a Student's t random variable with degree of freedom ν and let $t_\nu^{-1}(\cdot)$ denote its inverse. Let $G_\nu(\cdot)$ denote the distribution function of $\sqrt{\nu/\chi_\nu^2}$ and let $G_\nu^{-1}(\cdot)$ denote its inverse. Let $z_i(u_i, s) = t_{\nu_i}^{-1}(u_i)/G_{\nu_i}^{-1}(s)$ for $i = 1, \ldots, p$. Luo and Shevchenko (2012) have defined the t copula with degrees of freedom $(\nu_1, \ldots, \nu_p)$ and variance-covariance matrix $\mathbf{\Sigma}$ as

$$C\left(u_1, \ldots, u_p\right) = \int_0^1 \Phi_{\mathbf{\Sigma}}\left(z_1\left(u_1, s\right), \ldots, z_1\left(u_1, s\right)\right) ds, \qquad (1)$$

where $\Phi_{\mathbf{\Sigma}}$ denotes distribution function of a p-variate normal random vector with zero means and variance-covariance matrix $\mathbf{\Sigma}$. Copulas of several multivariate t distributions are particular cases of (1).

2.3 FGM copulas

Farlie-Gumbel-Morgenstern copula (Morgenstern, 1956; Nelson, 2006) is defined by

$$C\left(u_1, u_2\right) = u_1 u_2 \left[1 + \phi \left(1 - u_1\right)\left(1 - u_2\right)\right]$$

for $-1 \leq \phi \leq 1$. Independence corresponds to $\theta = 0$. The p-variate version is

$$C\left(u_1, \ldots, u_p\right) = u_1 \cdots u_p \left[1 + \sum_{k=2}^{p} \sum_{1 \leq j_1 < \cdots < j_k \leq p} \theta_{j_1, \ldots, j_k} \left(1 - u_{j_1}\right) \cdots \left(1 - u_{j_k}\right)\right]$$

for $-1 \leq \theta_{j_1, \ldots, j_k} \leq 1$ for all $j_1, \ldots, j_k$.

Several extensions of the Farlie-Gumbel-Morgenstern copula exist in the literature. Cambanis (1977) has proposed the extension

$$C\left(u_1, \ldots, u_p\right) = \prod_{k=1}^{p} u_k \left[1 + \sum_{c=2}^{p} \sum_{1 \leq i_2 < \cdots < i_c \leq p} a_{i_1, \ldots, i_c} \left(1 - u_{i_1}\right) \cdots \left(1 - u_{i_c}\right)\right]$$

for $-\infty < a_{i_1, \ldots, i_c} < \infty$ such that

$$\sum_{c=2}^{p} \sum_{1 \leq i_2 < \cdots < i_c \leq p} a_{i_1, \ldots, i_c} \delta_{i_1} \cdots \delta_{i_c} \geq -1$$

for all $-1 \leq \delta_i \leq 1$, $i = 1, \ldots, p$. An extension proposed by Ibragimov (2009) is

$$C\left(u_1, \ldots, u_p\right) = \prod_{i=1}^{p} u_i \left[1 + \sum_{c=2}^{p} \sum_{1 \leq i_2 < \cdots < i_c \leq p} a_{i_1, \ldots, i_c} \left(u_{i_1}^{\ell} - u_{i_1}^{\ell+1}\right) \cdots \left(u_{i_c}^{\ell} - u_{i_c}^{\ell+1}\right)\right]$$

for $-\infty < a_{i_1, \ldots, i_c} < \infty$ such that

$$\sum_{c=2}^{p} \sum_{1 \leq i_2 < \cdots < i_c \leq p} \left|a_{i_1, \ldots, i_c}\right| \leq 1.$$

Two extensions proposed by Bekrizadeh *et al.* (2012) are

$$C\left(u_1, u_2\right) = u_1 u_2 \left[1 + \theta \left(1 - u_1^{\alpha}\right)\left(1 - u_2^{\alpha}\right)\right]^n$$

and

$$C\left(u_1, \ldots, u_p\right) = \prod_{i=1}^{p} u_i \left\{1 + \sum_{k=2}^{p} \sum_{1 \leq j_1 < \cdots j_k \leq p} \theta_{j_1, \ldots, j_k} \left(1 - u_{j_1}^{\theta_{j_1, \ldots, j_k}}\right) \cdots \left(1 - u_{j_k}^{\theta_{j_1, \ldots, j_k}}\right)\right\}^n$$

for $\alpha > 0$, $\alpha_{j_1, \ldots, j_k} > 0$ and $n \geq 0$,

2.4 Fréchet's copula

Fréchet copula due to Fréchet (1958) is defined by

$$C(u_1, u_2) = a \min(u_1, u_2) + (1 - a - b) u_1 u_2 + b \max(u_1 + u_2 - 1, 0)$$

for $0 \leq a, b \leq 1$ and $a + b \leq 1$. Independence corresponds to $a = b = 0$. Complete dependence corresponds to $a = 1$.

A related copula due to Mardia (1970) is defined by

$$C(u_1, u_2) = \frac{\theta^2(1+\theta)}{2} \min(u_1, u_2) + \left(1 - \theta^2\right) u_1 u_2 + \frac{\theta^2(1-\theta)}{2} \max(u_1 + u_2 - 1, 0)$$

for $-1 \leq \theta \leq 1$. Another related copula considered by Gijbels *et al.* (2010) is

$$C(u_1, u_2) = \frac{\gamma\theta^2(1+\theta)}{2} \min(u_1, u_2) + \left(1 - \gamma\theta^2\right) u_1 u_2 + \frac{\gamma\theta^2(1-\theta)}{2} \max(u_1 + u_2 - 1, 0)$$

for $-1 \leq \theta \leq 1$ and $\gamma \leq 1/\theta^2$.

2.5 Gumbel's copula

The Gumbel-Barnett copula due to Gumbel (1960) and Barnett (1980) is defined by

$$C(u_1, u_2) = u_1 + u_2 - 1 + (1 - u_1)(1 - u_2) \exp\left[-\phi \log(1 - u_1) \log(1 - u_2)\right]$$

for $0 \leq \phi \leq 1$. Independence corresponds to $\phi = 0$. Another copula also due to Gumbel (1960) and Hougaard (1986) is

$$C(u_1, u_2) = \exp\left\{-\left[(-\log u_1)^\phi + (-\log u_2)^\phi\right]^{\frac{1}{\phi}}\right\}$$

for $\phi \geq 1$. Now independence corresponds to $\phi = 1$.

2.6 Plackett's copula

Plackett (1965) has defined the copula

$$C(u_1, u_2) = \frac{1 + (\theta - 1)(u_1 + u_2) - \sqrt{\left[1 + (\theta - 1)(u_1 + u_2)\right]^2 - 4\theta(\theta - 1) u_1 u_2}}{2(\theta - 1)}$$

for $\theta > 0$. Independence corresponds to $\theta = 1$.

2.7 Rodriguez-Lallena and Úbeda-Flores's copula

Sarmanov (1966) and Rodriguez-Lallena and Úbeda-Flores (2004) have defined the copula

$$C(u_1, u_2) = u_1 u_2 + \theta f(u_1) g(u_2) \tag{2}$$

for $0 \leq \theta \leq 1$ and $f, g : [0,1] \rightarrow \mathbb{R}$ such that i) $f(0) = f(1) = g(0) = g(1) = 0$; ii) f and g are absolutely continuous; iii) $\min(\alpha\delta, \beta\gamma) \geq -1$, where $\alpha = \inf\left\{ f^{'}(u) : u \in A \right\} < 0$, $\beta = \sup\left\{ f^{'}(u) : u \in A \right\} > 0$, $\gamma = \inf\left\{ g^{'}(v) : v \in B \right\} < 0$, $\delta = \sup\left\{ g^{'}(v) : v \in B \right\} > 0$, $A = \left\{ 0 \leq u \leq 1 : f^{'}(u) \text{ exists } \right\}$ and $B = \left\{ 0 \leq v \leq 1 : g^{'}(v) \text{ exists } \right\}$.

Many authors have studied special cases of (2) in detail. A special case given in Rodriguez-Lallena and Úbeda-Flores (2004) is

$$C\left(u_1, u_2\right) = u_1 u_2 + \theta u_1^a u_2^b \left(1 - u_1\right)^c \left(1 - u_2\right)^d$$

for $a, b, c, d \geq 1$. Rodriguez-Lallena and Úbeda-Flores (2004) have shown that this is a copula if and only if

$$-\frac{1}{\max\left(\nu\gamma, \omega\delta\right)} \leq \theta \leq -\frac{1}{\min\left(\nu\delta, \omega\gamma\right)},$$

where $\omega = -\nu = 1$ if $a = c = 1$, $\delta = -\gamma = 1$ if $b = d = 1$ and

$$\nu = -\left(\frac{a}{a+c}\right)^{a-1} \left[1 + \sqrt{\frac{c}{a(a+c-1)}}\right]^{a-1} \left(\frac{c}{a+c}\right)^{c-1} \left[1 - \sqrt{\frac{a}{c(a+c-1)}}\right]^{c-1} \sqrt{\frac{ac}{a+c-1}},$$

$$\omega = \left(\frac{a}{a+c}\right)^{a-1} \left[1 - \sqrt{\frac{c}{a(a+c-1)}}\right]^{a-1} \left(\frac{c}{a+c}\right)^{c-1} \left[1 + \sqrt{\frac{a}{c(a+c-1)}}\right]^{c-1} \sqrt{\frac{ac}{a+c-1}},$$

$$\gamma = -\left(\frac{b}{b+d}\right)^{b-1} \left[1 + \sqrt{\frac{d}{b(b+d-1)}}\right]^{b-1} \left(\frac{d}{b+d}\right)^{d-1} \left[1 - \sqrt{\frac{b}{d(b+d-1)}}\right]^{d-1} \sqrt{\frac{bd}{b+d-1}},$$

$$\delta = \left(\frac{b}{b+d}\right)^{b-1} \left[1 - \sqrt{\frac{d}{b(b+d-1)}}\right]^{b-1} \left(\frac{d}{b+d}\right)^{d-1} \left[1 + \sqrt{\frac{b}{d(b+d-1)}}\right]^{d-1} \sqrt{\frac{bd}{b+d-1}}.$$

Special cases also include the FGM copulas discussed in Section 2.3. Others include

$$C\left(u_1, u_2\right) = u_1 u_2 + \theta u_1 u_2 \left(1 - u_1\right)^{\gamma} \left(1 - u_2\right)^{\gamma}$$

due to Huang and Kotz (1999) for $0 \leq \theta \leq 1$ and $\gamma \geq 1$;

$$C\left(u_1, u_2\right) = u_1 u_2 + \theta u_1 u_2 \left(1 - u_1^{\gamma}\right) \left(1 - u_2^{\gamma}\right)$$

due to Huang and Kotz (1999) for $0 \leq \theta \leq 1$ and $\gamma \geq 1/2$;

$$C\left(u_1, u_2\right) = u_1 u_2 + \theta u_1^p u_2^p \left(1 - u_1\right)^q \left(1 - u_2\right)^q$$

due to Lai and Xie (2000) for $0 \leq \theta \leq 1$;

$$C\left(u_1, u_2\right) = u_1^p u_2^p \left[1 + \theta \left(1 - u_1^q\right)^n \left(1 - u_2^q\right)^n\right]$$

due to Bairamov and Kotz (2002) for $0 \leq \theta \leq 1$;

$$C\left(u_1, u_2\right) = \theta u_1^{\alpha} u_2^{\beta} \left(1 - u_1\right) \left(1 - u_2\right)$$

and

$$C\left(u_1, u_2\right) = \theta u_1 u_2 \left(1 - u_1\right)^{\alpha} \left(1 - u_2\right)^{\beta}$$

due to Jung *et al.* (2007) for $\alpha \geq 1$, $\beta \geq 1$ and $-1 \leq \theta \leq 1$.

Based on (2), Kim *et al.* (2009) have introduced the three-dimensional FGM copula

$$\begin{aligned} C\left(u_1, u_2, u_3\right) &= u_1 u_2 u_3 \left[1 + \theta_{13}\left(1 - u_1\right)\left(1 - u_3\right)\right]\left[1 + \theta_{23}\left(1 - u_2\right)\left(1 - u_3\right)\right] \\ & \quad \cdot \left[1 + \theta_{12}\left(1 - u_1\right)\left(1 - u_2\right)\right]\left[1 + \theta_{13} u_1\left(1 - u_3\right)\right]\left[1 + \theta_{23} u_2\left(1 - u_3\right)\right]. \end{aligned}$$

2.8 Marshall and Olkin's copula

Marshall and Olkin (1967) have defined the copula

$$C\left(u_1, u_2\right) = \begin{cases} u_1^{1-\alpha} u_2, & \text{if } u_1^\alpha \geq u_2^\beta, \\ u_1 u_2^{1-\beta}, & \text{if } u_1^\alpha < u_2^\beta \end{cases}$$

for $0 \leq \alpha, \beta \leq 1$. Independence corresponds to $\alpha = \beta = 0$. Complete dependence corresponds to $\alpha = \beta = 1$.

2.9 Galambos's copula

Galambos (1975) has defined the copula

$$C\left(u_1, u_2\right) = u_1 u_2 \exp\left\{\left[\left(1 - u_1\right)^{-\theta} + \left(1 - u_2\right)^{-\theta}\right]^{-1/\theta}\right\}$$

for $\theta \geq 0$. A p-variate version is

$$C\left(u_1, \ldots, u_p\right) = \exp\left\{\sum_{S \in \mathcal{S}} (-1)^{|S|}\left[\sum_{i \in S}\left(1 - u_i\right)^{-\theta}\right]^{-1/\theta}\right\}$$

for $\theta \geq 0$, where $\mathcal{S}$ is the set of all nonempty subsets of $\{1, \ldots, p\}$. Independence corresponds to $\theta = 0$.

2.10 AMH copula

The Ali-Mikhail-Haq copula due to Ali *et al.* (1978) is defined by

$$C\left(u_1, \ldots, u_p\right) = (1 - \alpha)\left[\prod_{i=1}^{p}\left(\frac{1 - \alpha}{u_i} + \alpha\right) - \alpha\right]^{-1}$$

for $-1 \leq \alpha \leq 1$. Independence corresponds to $\alpha = 0$.

2.11 Clayton's copula

Clayton (1978), Cook and Johnson (1981) and Oakes (1982) have defined the copula

$$C\left(u_1, \ldots, u_p\right) = \left[\sum_{i=1}^{p} u_i^{-\alpha} - p + 1\right]^{-1/\alpha}$$

for $\alpha \geq 0$. Independence corresponds to $\alpha = 0$. Complete dependence corresponds to $\alpha = \infty$.

2.12 Frank's copula

Frank (1979) has defined the copula

$$C\left(u_1, u_2\right) = \log_\alpha \left[1 + \frac{\left(\alpha^{u_1} - 1\right)\left(\alpha^{u_2} - 1\right)}{\alpha - 1}\right]$$

for $\alpha \geq 0$. Independence corresponds to $\alpha = 1$. The p-variate version is

$$C\left(u_1, \ldots, u_p\right) = \log_\alpha \left[1 + \frac{\displaystyle\prod_{i=1}^{p}\left(\alpha^{u_i} - 1\right)}{(\alpha - 1)^{p-1}}\right]$$

for $\alpha \geq 0$.

2.13 Cuadras and Augé's copula

Cuadras and Augé (1981) have defined the copula

$$C\left(u_1, u_2\right) = \left[\min\left(u_1, u_2\right)\right]^\theta \left(u_1 u_2\right)^{1-\theta} \tag{3}$$

for $0 \leq \theta \leq 1$. Independence corresponds to $\theta = 0$. Complete dependence corresponds to $\theta = 1$.

A p-variate version of (3) due to Cuadras (2009) is

$$C\left(u_1, \ldots, u_p\right) = \min\left(u_1, \ldots, u_p\right) \prod_{i=2}^{p} u_{(i)}^{\prod_{j=1}^{i-1}\left(1-\theta_{ij}\right)}$$

for $0 \leq \theta_{ij} \leq 1$, where $u_{(1)} \leq \cdots \leq u_{(p)}$ are the sorted values of $u_1, \ldots, u_p$. A more general version is

$$C\left(u_1, \ldots, u_p\right) = \min\left(u_1, \ldots, u_p\right) \prod_{i=2}^{p} u_{(i)}^{\alpha_i}$$

for some suitable α_i, see Cuadras (2009).

2.14 Extreme value copula

Let $A : [0,1] \rightarrow [1/2,1]$ be a convex function satisfying $\max(w, 1-w) \leq A(w) \leq 1$ for all $w \in [0,1]$. The extreme value copula due to Pickands (1981) is defined by

$$C\left(u_1, u_2\right) = \exp\left[\log\left(u_1 u_2\right) A\left(\frac{\log u_2}{\log\left(u_1 u_2\right)}\right)\right]. \tag{4}$$

Independence corresponds to $A(w) = 1$ for all $w \in [0,1]$. Complete dependence corresponds to $A(w) = \max(w, 1-w)$. Some popular models for $A(\cdot)$ are

$$A(w) = \left[w^\theta + (1-w)^\theta\right]^{1/\theta}$$

due to Gumbel (1960), where $\theta \geq 1$ (see Section 2.5);

$$A(w) = 1 - \left[w^{-\theta} + (1-w)^{-\theta}\right]^{-1/\theta}$$

due to Galambos (1975), where $\theta \geq 0$;

$$A(w) = 1 - (\theta + \phi) w + \theta w^2 + \phi w^3$$

due to Tawn (1988), where $\theta \geq 0$, $\theta + 3\phi \geq 0$, $\theta + \phi \leq 1$ and $\theta + 2\phi \leq 1$;

$$A(w) = \left(1 - \phi_1\right)(1-w) + \left(1 - \phi_2\right) w + \left[\left(\phi_1 w\right)^{1/\theta} + \left(\phi_2 (1-w)\right)^{1/\theta}\right]^\theta$$

due to Tawn (1988), where $0 < \theta \leq 1$ and $0 \leq \phi_1, \phi_2 \leq 1$;

$$A(w) = w \Phi\left(\frac{1}{\theta} + \frac{\theta}{2} \log \frac{w}{1-w}\right) + (1-w) \Phi\left(\frac{1}{\theta} - \frac{\theta}{2} \log \frac{w}{1-w}\right)$$

due to Hüsler and Reiss (1989), where $\theta \geq 0$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable;

$$A(w) = 1 - \left[\left(\phi_1 (1-w)\right)^{-1/\theta} + \left(\phi_2 w\right)^{-1/\theta}\right]^{-\theta}$$

due to Joe (1990), where $\theta > 0$ and $0 \leq \phi_1, \phi_2 \leq 1$;

$$A(w) = \int_0^1 \max\left[(1-\beta)(1-w) t^{-\beta}, (1-\delta) w (1-t)^{-\delta}\right] dt$$

due to Joe *et al.* (1992) and Coles and Tawn (1994), where $(\beta, \delta) \in (0,1)^2 \cup (-\infty, 0)^2$; and,

$$A(w) = w t_{\chi+1}\left(\sqrt{\frac{1+\chi}{1-\rho^2}}\left[\left(\frac{w}{1-w}\right)^{1/\xi} - \rho\right]\right) + (1-w) t_{\chi+1}\left(\sqrt{\frac{1+\chi}{1-\rho^2}}\left[\left(\frac{1-w}{w}\right)^{1/\xi} - \rho\right]\right)$$

due to Demarta and McNeil (2005), where $-1 < \rho < 1$, $\xi > 0$ and $t_\nu(\cdot)$ is the cumulative distribution function of a Student's t random variable with ν degrees of freedom.

The p-variate generalization of (4) is

$$C(u_1, \ldots, u_p) = \exp\left[\sum_{i=1}^{p} \log u_i A\left(\frac{\log u_1}{\sum_{i=1}^{p} \log u_i}, \ldots, \frac{\log u_{p-1}}{\sum_{i=1}^{p} \log u_i}\right)\right],$$

where $A(\cdot)$ is a convex function on the $(p-1)$-dimensional simplex satisfying $\max(w_1, \ldots, w_p) \leq A(w_1, \ldots, w_p) \leq 1$ for all $(w_1, \ldots, w_p)$ in the $(p-1)$-dimensional simplex. Models for $A(\cdot)$ follow easily from the ones given for the bivariate version. A p-version of the model for $A(\cdot)$ due to Demarta and McNeil (2005) is given in Nikoloulopoulos *et al.* (2009).

2.15 Archimedean copula

The Archimedean copula due to Genest and MacKay (1986) is defined by

$$C(u_1, \ldots, u_p) = \psi\left(\sum_{i=1}^{p} \psi^{-1}(u_i)\right),$$

where $\psi : [0, 1] \to [0, \infty)$ is a real valued function satisfying $(-1)^k d^k \psi(x)/d^k x \geq 0$ for all $x \geq 0$ and $k = 1, \ldots, p-2$ and $(-1)^{p-2} \psi^{p-2}(x)$ is non-increasing and convex. Particular cases of this copula include the Plackett's copula in Section 2.6, the Nelson's copula in Section 2.43, the AMH copula in Section 2.10, the Clayton's copula in Section 2.11 and the Frank copula in Section 2.12.

Extensions of the Archimedean copula include nested Archimedean copulas studied by Joe (1997), Whelan (2004), McNeil *et al.* (2005), Hofert (2008), McNeil (2008) and Savu and Trede (2010). An asymmetric Archimedean copula due to Wei and Hu (2002) is

$$C(u_1, \ldots, u_p) = \psi\left(\psi^{-1} \circ \phi\left(\sum_{i=1}^{k} \psi^{-1}(u_i)\right) + \sum_{i=k+1}^{p} \psi^{-1}(u_i)\right)$$

for $2 \leq k \leq p$, where $\phi : [0, 1] \to [0, \infty)$ satisfies the same properties as ψ. An extension due to Durante *et al.* (2007a) is

$$C(u_1, u_2) = \phi^{-1}(\phi(\min(u_1, u_2)) + \psi(\max(u_1, u_2))),$$

where $\phi : [0, 1] \to [0, \infty)$ is continuous, strictly decreasing, convex and $\psi : [0, 1] \to [0, \infty)$ is continuous, decreasing such that $\psi(1) = 0$ and $\psi - \phi$ is increasing. Another extension due to Durante *et al.* (2007a) is

$$C(u_1, u_2) = \phi^{-1}(\phi(\min(u_1, u_2))\,\psi(\max(u_1, u_2))),$$

where $\phi : [0, 1] \to [0, 1]$ is continuous, increasing, log-concave and $\psi : [0, 1] \to [0, 1]$ is continuous, increasing such that $\psi(1) = 1$ and ϕ/ψ is increasing.

A related copula given in Joe (1997) is the following: if ϕ denotes a Laplace transform, ϕ^{-1} its inverse and $C^{'}$ a given copula then

$$C(u_1, u_2) = \phi\left(-\log C^{'}\left(\exp\left[-\phi^{-1}(u_1)\right], \exp\left[-\phi^{-1}(u_2)\right]\right)\right)$$

is a valid copula.

2.16 Alsina *et al.*'s copula

For given copulas D, P and given functions $f, \alpha_1, \ldots, \alpha_p : [0,1]^p \to [0,1]$, Alsina *et al.* (1991) have shown that

$$C(u_1, \ldots, u_p) = \theta P(u_1, \ldots, u_p) + (1-\theta)\left[f(u_1, \ldots, u_p) + D(\alpha_1(u_1, \ldots, u_p), \ldots, \alpha_p(u_1, \ldots, u_p))\right]$$

is a copula for $0 \leq \theta \leq 1$ if certain conditions are satisfied, see Theorem 3.4 in Alsina *et al.* (1991).

2.17 Raftery copula

Nelsen (1991) has introduced the copula defined by

$$C(u_1, u_2) = \begin{cases} u_1 - \dfrac{1-\theta}{1+\theta} u_1^{\frac{1}{1-\theta}} \left(u_2^{-\frac{\theta}{1-\theta}} - u_2^{\frac{1}{1-\theta}} \right), & \text{if } u_1 \leq u_2, \\ u_2 - \dfrac{1-\theta}{1+\theta} u_2^{\frac{1}{1-\theta}} \left(u_1^{-\frac{\theta}{1-\theta}} - u_1^{\frac{1}{1-\theta}} \right), & \text{if } u_1 > u_2 \end{cases}$$

for $0 \leq \theta < 1$. Complete dependence corresponds to $\theta = 0$.

2.18 Brownian motion copula

The Brownian motion copula due to Darsow *et al.* (1992) is defined by

$$C(u_1, u_2) = \int_0^{u_1} \Phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2) - \sqrt{s}\Phi^{-1}(x)}{\sqrt{t-s}} \right) dx$$

for $t > s$, where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

2.19 Koehler and Symanowski's copula

Let $V = \{1, \ldots, p\}$ denote an index set, let $\mathcal{V}$ denote the power set of V, and let $\mathcal{I}$ denote the set of all $I \in \mathcal{V}$ with $|I| \geq 2$. For all subsets $I \in \mathcal{I}$, let $\alpha_I > 0$ and $\alpha_i > 0$ for all $i \in V$ be such that $\alpha_{i+} = \alpha_i + \sum_{I \in \mathcal{I}} \alpha_I > 0$ for all $i \in I$. Koehler and Symanowski (1995) have shown that

$$C(u_1, \ldots, u_p) = \frac{\displaystyle\prod_{i \in V} u_i}{\displaystyle\prod_{I \in \mathcal{I}} \left[\sum_{i \in I} \prod_{j \in I, j \neq i} u_j^{\alpha_{j+}} - (\mid I \mid -1) \prod_{i \in I} u_i^{\alpha_{i+}} \right]^{\alpha_I}}$$

is a valid copula.

2.20 Quesada-Molina and Rodríguez-Lallena's copulas

Let $a, b, c : [0, 1] \to \mathbb{R}$. Quesada-Molina and Rodríguez-Lallena (1995) have shown that

$$C(u_1, u_2) = a(u_2) u_1^2 + b(u_2) u_1 + c(u_2)$$

is a copula if and only if $b(u_2) = u_2 - a(u_2)$ and $c(u_2) = 0$ for all $0 \leq u_2 \leq 1$, $a(0) = a(1) = 0$, and $|a(u_1) - a(u_2)| \leq |u_1 - u_2|$ for all $0 \leq u_1, u_2 \leq 1$. Quesada-Molina and Rodríguez-Lallena (1995) have also shown that

$$C(u_1, u_2) = u_1 u_2 + u_1 (1 - u_1) a(u_2)$$

is a copula if and only if $a(u)$ is an absolutely continuous function, $a^{'}(u) \leq 1$ almost everywhere in $[0, 1]$, and $\left| a^{'}(u) \right| < \min(u, 1 - u)$ for all $0 \leq u \leq 1$.

2.21 Shih and Louis's copula

Shih and Louis (1995) have defined the copula

$$C(u_1, u_2) = \begin{cases} (1 - \rho) u_1 u_2 + \rho \min(u_1, u_2), & \text{if } \rho > 0, \\ \\ (1 + \rho) u_1 u_2 + \rho (u_1 - 1 + u_2) \Theta(u_1 - 1 + u_2), & \text{if } \rho \leq 0, \end{cases}$$

where $\Theta(a) = 1$ if $a \geq 0$ and $\Theta(a) = 0$ if $a < 0$. Independence corresponds to $\rho = 0$. Complete dependence corresponds to $\rho = 1$.

2.22 Joe's copulas

Joe and Hu (1996) have proposed several copulas. Here, we discuss three of them. The first of these is defined by

$$C(u_1, u_2) = \left\{ 1 + \left[\left(u_1^{-a} - 1 \right)^b + \left(u_2^{-a} - 1 \right)^b \right]^{\frac{1}{b}} \right\}^{-\frac{1}{a}}$$

for $a > 0$ and $b \geq 1$. The second is defined by

$$C(u_1, u_2) = \left\{ u_1^{-a} + u_2^{-a} - 1 - \left[\left(u_1^{-a} - 1 \right)^{-b} + \left(u_2^{-a} - 1 \right)^{-b} \right]^{-\frac{1}{b}} \right\}^{-\frac{1}{a}}$$

for $a \geq 0$ and $b > 0$. The third (see also Joe (1997, page 153)) is defined by

$$C(u_1, u_2) = 1 - \left\{ 1 - \left[\left(1 - u_1^{-a} \right)^{-b} + \left(1 - u_2^{-a} \right)^{-b} - 1 \right]^{-\frac{1}{b}} \right\}^{\frac{1}{a}}$$

for $a \geq 1$ and $b > 0$. Another copula presented in Joe (1997) is

$$C(u_1, u_2) = \exp \left\{ - \left[\theta_2^{-1} \log \left(\exp \left(-\theta_2 (\log u_1)^{\theta_1} \right) + \exp \left(-\theta_2 (\log u_2)^{\theta_1} \right) - 1 \right) \right]^{\frac{1}{\theta_1}} \right\}$$

for $\theta_1 \geq 1$ and $\theta_2 \geq 1$.

2.23 Marshall's copula

Suppose $f, g : [0,1] \to [0,1]$ are continuous and increasing functions such that $f(0) = g(0) = 0$, $f(1) = g(1) = 1$ and both $f(t)/t$ and $g(t)/t$ are decreasing. With this assumption, Marshall (1996) has shown that

$$C(u_1, u_2) = u_1 u_2 \min\left[\frac{f(u_1)}{u_1}, \frac{g(u_2)}{u_2}\right]$$

is a valid copula.

2.24 Fredricks and Nelsen's copula

Let $f : [0,1] \to [0,1]$ be such that $f(1) = 1$, $\mid f(u_1) - f(u_2) \mid \leq 2 \mid u_1 - u_2 \mid$ for all $u_1, u_2 \in [0,1]$, and $f(u) \leq u$ for all $u \in [0,1]$. Fredricks and Nelsen (1997) have shown that

$$C(u_1, u_2) = \min\left[u_1, u_2, \frac{f(u_1) + f(u_2)}{2}\right]$$

is a valid copula. In a subsequent development, Wysocki (2012) has shown that

$$C(u_1, u_2) = \lim_{k \to \infty} f^k \left[f^{-k}(u_1) + f^{-k}(u_2) - 1\right]$$

is a valid copula.

2.25 Bertino copulas

Let $\delta : [0,1] \to [0,1]$ be such that $\delta(0) = 0$, $\delta(1) = 1$, $\delta(v) \leq v$ for all $0 \leq v \leq 1$, and $0 \leq \delta(v) - \delta(w) \leq 2(v - w)$ for all $0 \leq v < w \leq 1$. Fredricks and Nelsen (1997) have shown that

$$C(u_1, u_2) = \min(u_1, u_2) - \min_{u_1 \leq w \leq u_2} [w - \delta(w)]$$

are valid copulas referred to as Bertino copulas.

2.26 Linear Spearman copula

Joe (1997, page 148) has defined the linear Spearman copula as that given by

$$C(u_1, u_2) = \begin{cases} [u_1 + \theta(1 - u_1)] u_2, & \text{if } u_2 \leq u_1,\ 0 \leq \theta \leq 1, \\ [u_2 + \theta(1 - u_2)] u_1, & \text{if } u_2 > u_1,\ 0 \leq \theta \leq 1, \\ (1 + \theta) u_1 u_2, & \text{if } u_1 + u_2 < 1,\ -1 \leq \theta \leq 0, \\ u_1 u_2 + \theta(1 - u_1)(1 - u_2), & \text{if } u_1 + u_2 \geq 1,\ -1 \leq \theta \leq 0. \end{cases}$$

Independence corresponds to $\theta = 0$. Complete dependence corresponds to $\theta = 1$.

2.27 Nelson *et al.*'s copulas

Let $\alpha, \beta : [0,1] \to \mathbb{R}$ with $\alpha(0)=\alpha(1)=\beta(0)=\beta(1)=0$. Nelson *et al.* (1997) have shown that

$$C(u_1,u_2) = u_1u_2 + u_1(1-u_1)\left[\alpha(u_2)(1-u_1)+\beta(u_2)u_1\right] \tag{5}$$

is a valid copula if and only if

$$\begin{aligned}&\left[(1-u_1)^2+(1-u_2)^2+u_1u_2-1\right]\frac{\alpha(v_2)-\alpha(v_1)}{v_2-v_1}\\ &-\left[u_1^2+u_2^2+(1-u_1)(1-u_2)-1\right]\frac{\beta(v_2)-\beta(v_1)}{v_2-v_1} \geq -1\end{aligned}$$

for every $0 \leq u_1 < u_2, v_1 < v_2 \leq 1$.

Nelson *et al.* (1997) have also shown that (5) is a copula if and only if $\alpha, \beta : [0,1] \to \mathbb{R}$ are absolutely continuous functions and $\left(\alpha'(u_2), \beta'(u_2)\right)$ lies in $\{(u_1,u_2) \in \mathbb{R}^2 : u_1^2 - u_1u_2 + u_2^2 - 3u_1 + 3u_2 = 0\}$ or $\{(u_1,u_2) \in \mathbb{R}^2 : -1 \leq u_1 \leq 2, -2 \leq u_2 \leq 1\}$. Nelson *et al.* (1997) have also shown that (5) is a copula if and only if $\alpha, \beta : [0,1] \to \mathbb{R}$ are absolutely continuous functions and $1-\alpha'(u_2)\left(1-4u_1+3u_1^2\right)+\beta'(u_2)\left(2u_1-3u_1^2\right) \geq 0$ for all $0 \leq u_1, u_2 \leq 1$. Nelson *et al.* (1997) have also shown that (5) is a copula if and only if $\max(-u, 3(u-1)) \leq \alpha(u) \leq \min(1-u, 3u)$ and $\max(-3u, u-1) \leq \beta(u) \leq \min(3(1-u), u)$ for $0 \leq u \leq 1$. Nelson *et al.* (1997) have also shown that

$$\begin{aligned}C(u_1,u_2) &= u_1u_2 + A_1u_1u_2^2(1-u_1)^2(1-u_2) + A_2u_1u_2(1-u_1)^2(1-u_2)^2\\ &\quad + B_1u_1^2u_2^2(1-u_1)(1-u_2) + B_2u_1^2u_2(1-u_1)(1-u_2)^2\end{aligned}$$

is a copula if A_1, A_2, B_1 and B_2 are in $\{(u_1,u_2) \in \mathbb{R}^2 : u_1^2 - u_1u_2 + u_2^2 - 3u_1 + 3u_2 = 0\}$ or $\{(u_1,u_2) \in \mathbb{R}^2 : -1 \leq u_1 \leq 2, -2 \leq u_2 \leq 1\}$. In particular,

$$C(u_1,u_2) = u_1u_2\left\{1+(1-u_1)(1-u_2)\left[a+b(1-2u_1)(1-2u_2)\right]\right\}$$

is a copula if either $-1 \leq b \leq 1/2$ and $|a| \leq b+1$ or $1/2 \leq b \leq 2$ and $|a| \leq \sqrt{6b-3b^2}$.

2.28 Archimax copula

Let $\phi(\cdot)$ be as defined in Section 2.15 and let $A(\cdot)$ be as defined in Section 2.14. Capéraà *et al.* (2000) have shown that

$$C(u_1,u_2) = \phi^{-1}\left(\min\left(\phi(0), \left[\phi(u_1)+\phi(u_2)\right]A\left(\frac{\phi(u_1)}{\phi(u_1)+\phi(u_2)}\right)\right)\right)$$

is a copula. It is referred to as the Archimax copula.

2.29 Cubic copula

Durrleman *et al.* (2000) have defined a copula referred to as a cubic copula by

$$C(u_1,u_2) = u_1u_2\left[1+\alpha(u_1-1)(u_2-1)(2u_1-1)(2u_2-1)\right]$$

for $-1 \leq \alpha \leq 2$. Independence corresponds to $\alpha = 0$.

2.30 Polynomial copula

A polynomial copula of degree m due to Drouet-Mari and Kotz (2001) is defined by

$$C(u_1, u_2) = u_1 u_2 \left[1 + \sum_{k \geq 1, q \geq 1, k+q \leq m-2} \frac{\theta_{kq}}{(k+1)(q+1)} \left(u_1^k - 1 \right) \left(u_2^q - 1 \right) \right]$$

for $k \geq 1$ and $q \geq 1$, where

$$0 \leq \min \left[\sum_{k \geq 1, q \geq 1} \frac{q \theta_{kq}}{(k+1)(q+1)}, \sum_{k \geq 1, q \geq 1} \frac{k \theta_{kq}}{(k+1)(q+1)} \right] \leq 1.$$

Independence corresponds to $\theta_{kq} = 0$ for all k and q.

2.31 Burr copulas

Frees and Valdez (2002) have defined what is referred to as a Burr copula as

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1 \right]^{-\alpha}$$

for $\alpha > 0$. An extension of this copula provided by de Waal and van Gelder (2005) is

$$\begin{aligned} C(u_1, u_2) &= u_1 + u_2 - 1 + \left[(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1 \right]^{-\alpha} \\ &\quad + \beta \left\{ \left[(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1 \right]^{-\alpha} \right. \\ &\quad + \left[2(1-u_1)^{-1/\alpha} + 2(1-u_2)^{-1/\alpha} - 3 \right]^{-\alpha} \\ &\quad + \left[2(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 2 \right]^{-\alpha} \\ &\quad \left. - \left[(1-u_1)^{-1/\alpha} + 2(1-u_2)^{-1/\alpha} - 2 \right]^{-\alpha} \right\} \end{aligned}$$

for $\alpha > 0$ and $-1 \leq \beta \leq 1$.

2.32 Knockaert's copula

Knockaert (2002) has defined a copula by

$$\begin{aligned} C(u_1, u_2) &= u_1 u_2 + \frac{\epsilon}{4\pi^2 mn} \Big\{ \cos\left[2\pi(m u_2 - \Delta)\right] + \cos\left[2\pi(n u_1 - \Delta)\right] \\ &\quad - \cos\left[2\pi(n u_1 + m u_2 - \Delta)\right] - \cos\left[2\pi \Delta\right] \Big\} \end{aligned}$$

for $\epsilon = -1, 1$, $0 \leq \Delta \leq 2\pi$ and $m, n = \ldots, -2, -1, 0, 1, 2, \ldots$. Independence corresponds to $m \to \infty$ or $n \to \infty$.

2.33 Lévy copula

Let $\nu(dx,dy)$ denote a bivariate Lévy measure. The Lévy copula due to Tankov (2003) is defined by

$$C(u_1,u_2) = U\left(U_1^{-1}(u_1), U_2^{-1}(u_2)\right),$$

where

$$U_1(a) = \int_0^\infty \int_a^\infty \nu(dx,dy),$$

$$U_2(b) = \int_b^\infty \int_0^\infty \nu(dx,dy),$$

and

$$U(a,b) = \int_b^\infty \int_a^\infty \nu(dx,dy).$$

2.34 Hürlimann's copula

Let $0 \leq \theta_{ij} \leq 1$ for $i = 1,\ldots,p$ and $j = 1,\ldots,p$. Hürlimann (2004) has proposed the copula

$$C(u_1,\ldots,u_p) = \frac{1}{c_p}\left\{ p\prod_{i=1}^p u_i + \sum_{r=2}^p \sum_{i_1 \neq \cdots = i_r} \left[\prod_{j=2}^r \frac{\theta_{i_1 i_j}}{1-\theta_{i_1 i_j}}\right] \min_{1\leq j\leq r} u_{i_j} \left[\prod_{k \in \{i_1,\ldots,i_r\}} u_k\right]\right\},$$

where

$$c_p = \sum_{i=1}^p \prod_{j\neq i} \frac{1}{1-\theta_{ij}}.$$

2.35 Kim and Sungur's copula

Let $f,g : [0,1] \to \mathbb{R}$ be non-zero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$. Kim and Sungur (2004) have shown that

$$C(u_1,u_2) = u_1u_2 + \theta f(u_1) g(u_2)$$

is a copula for $-1/\max(\alpha\gamma,\beta\delta) \leq \theta \leq -1/\min(\alpha\delta,\beta\gamma)$ and $\min(\alpha\delta,\beta\gamma) \geq -1$, where $\alpha = \inf\left\{f^{'}(u) : u \in A\right\}$, $\beta = \sup\left\{f^{'}(u) : u \in A\right\}$, $\gamma = \inf\left\{g^{'}(v) : v \in B\right\}$, $\delta = \sup\left\{g^{'}(v) : v \in B\right\}$, $A = \left\{0 \leq u \leq 1 : f^{'}(u) \text{ exists }\right\}$ and $B = \left\{0 \leq v \leq 1 : g^{'}(v) \text{ exists }\right\}$.

2.36 Bernstein copulas

Let $\alpha\left(k_1/m_1, \ldots, k_p/m_p\right)$ be real constants for $1 \leq k_i \leq m_i$ and $i = 1, \ldots, p$. Also let

$$P_{k_i,m_i}\left(u_i\right) = \binom{m_i}{k_i} u_i^{k_i}\left(1-u_i\right)^{m_i-k_i}$$

for $i = 1, \ldots, p$. Sancetta and Satchell (2004) have defined Bernstein copula as

$$\sum_{k_1=1}^{m_1} \cdots \sum_{k_p=1}^{m_p} \alpha\left(\frac{k_1}{m_1}, \ldots, \frac{k_p}{m_p}\right) P_{k_1,m_1}\left(u_1\right) \cdots P_{k_p,m_p}\left(u_p\right)$$

provided that

$$\sum_{\ell_1=1}^{1} \cdots \sum_{\ell_p=0}^{1} (-1)^{\ell_1+\cdots+\ell_p} \alpha\left(\frac{k_1+\ell_1}{m_1}, \ldots, \frac{k_p+\ell_p}{m_p}\right) < 1$$

for $0 \leq k_i \leq m_i$, $i = 1, \ldots, p$, and

$$\min\left(\frac{k_1}{m_1} + \cdots + \frac{k_p}{m_p} - p + 1\right) \leq \alpha\left(\frac{k_1}{m_1}, \ldots, \frac{k_p}{m_p}\right) \leq \min\left(\frac{k_1}{m_1}, \ldots, \frac{k_p}{m_p}\right).$$

2.37 Order statistics copula

There are several order statistics copulas. One due to Schmitz (2004) is

$$C\left(u_1, u_2\right) = u_2 - \left[\left(1-u_1\right)^{1/n} + u_2^{1/n}\right]^n$$

for n an integer greater than or equal to one. This copula is related to the Clayton copula, see Section 2.11.

2.38 Power variance copula

Andersen (2005) and Massonnet *et al.* (2009) have introduced the power variance copula as that defined by

$$C\left(u_1, u_2\right) = \exp\left[\frac{v}{\theta(1-v)}\left[1 - \left\{\sum_{j=1}^{4}\left(1+\theta\left(1-\frac{1}{v}\right)\log u_j\right)^{\frac{1}{1-v}} - 3\right\}^{1-v}\right]\right]$$

for $\theta \geq 0$ and $0 \leq v \geq 1$. The particular case for $v \to 1$ is the inverse Gaussian copula defined by

$$C\left(u_1, u_2\right) = \exp\left[\frac{1}{\theta} - \left[\frac{1}{\theta} + \sum_{j=1}^{4} \log u_j \left\{\log u_j - \frac{2}{\theta}\right\}\right]^{1/2}\right]$$

for $\theta \geq 0$.

2.39 Durante and Sempi's copulas

Let $\phi : [0,1] \to [0,1]$ be a continuous, strictly increasing and concave function with $\phi(1) = 1$. For any given copula C, Durante and Sempi (2005) have shown that

$$C(u_1, u_2) = \phi^{-1}\left(\max\left(\phi(0), C\left(\phi(u_1), \phi(u_2)\right)\right)\right)$$

is also a copula.

2.40 Dolati and Úbeda-Flores's copula

Let C_{ij}, $1 \leq i < j \leq p$ be given two-dimensional copulas. Dolati and Úbeda-Flores (2006) have shown that

$$C(u_1, \ldots, u_p) = \sum_{1 \leq i < j \leq p} C_{ij}(u_i, u_j) \prod_{k=1, k \neq i,j}^{p} u_k - \frac{(p-2)(p+1)}{2} \prod_{i=1}^{p} u_i$$

is a copula if and only if

$$\sum_{1 \leq i < j \leq p} \frac{C_{ij}(v_i, v_j) - C_{ij}(v_i, u_j) - C_{ij}(u_i, v_j) + C_{ij}(v_i, v_j)}{(v_i - u_i)(v_j - u_j)} \geq \frac{(p-2)(p+1)}{2}$$

for $0 \leq u_k < v_k \leq 1$ and $k = 1, \ldots, p$.

2.41 Fischer and Hinzmann's copulas

Fischer and Hinzmann (2006) have defined a copula by

$$C(u_1, u_2) = \left\{\alpha\left[\min(u_1, u_2)\right]^m + (1-\alpha)\left[u_1 u_2\right]^m\right\}^{1/m}$$

for $0 \leq \alpha \leq 1$ and $-\infty < m < \infty$. Independence corresponds to $\alpha = 0$ and $m = 1$. Complete dependence corresponds to $\alpha = 1$ and $m = 1$.

2.42 Kallsen and Tankov's copulas

Let $\phi : [-1,1] \to [-\infty, \infty]$ be a strictly increasing continuous function with $\phi(1) = \infty$, $\phi(0) = 0$ and $\phi(-1) = -\infty$, having derivatives of orders up to p on $(-1,0)$ and $(0,1)$, and satisfying $d^p \phi(\exp(x)) / dx^p \geq 0$ and $d^p \phi(-\exp(x)) / dx^p \leq 0$ for $-\infty < x < 0$. Let

$$\psi(u) = 2^{p-2}\left\{\phi(u) - \phi(-u)\right\}$$

for $-1 \leq u \leq 1$. With this notation, Kallsen and Tankov (2006) have shown that

$$C(u_1, \ldots, u_p) = \phi\left(\prod_{i=1}^{p} \psi^{-1}(u_i)\right)$$

is a valid copula.

2.43 Nelson's copulas

Nelson (2006) has presented a range of different copulas. Some of them are

$$C(u_1, \ldots, u_p) = \exp\left[1 - \left\{1 + \sum_{i=1}^{p} \left[(1 - \log u_i)^\theta - 1\right]\right\}^{\frac{1}{\theta}}\right]$$

for $\theta \geq 1$ with independence corresponding to $\theta = 1$;

$$C(u_1, u_2) = \left[1 + \frac{\left[(1+u_1)^{-\alpha} - 1\right]\left[(1+u_2)^{-\alpha} - 1\right]}{2^{-\alpha} - 1}\right]^{-1/\alpha} - 1$$

for $\alpha > 0$;

$$C(u_1, u_2) = -\frac{1}{\theta} \log\left\{1 + \frac{[\exp(-\theta u_1) - 1][\exp(-\theta u_2) - 1]}{\exp(-\theta) - 1}\right\}$$

for $-\infty < \theta < \infty$ with independence corresponding to $\theta \to 0$;

$$C(u_1, u_2) = \frac{\theta^2 u_1 u_2 + (1 - u_1)(1 - u_2)}{\theta^2 + (\theta - 1)^2 (1 - u_1)(1 - u_2)}$$

for $1 \leq \theta < \infty$;

$$C(u_1, u_2) = u_1 u_2 \exp\left[-\theta \log u_1 \log u_2\right]$$

for $0 < \theta \leq 1$ with independence corresponding to $\theta \to 0$;

$$C(u_1, u_2) = \frac{\theta^2 u_1 u_2}{\left[1 - (1 - u_1)^\theta (1 - u_2)^\theta\right]^{1/\theta}}$$

for $0 < \theta \leq 1$;

$$C(u_1, u_2) = \frac{1}{2}\left[S + \sqrt{S^2 + 4\theta}\right]$$

for $0 \leq \theta < \infty$, where

$$S = u_1 + u_2 - \frac{1}{u_1} - \frac{1}{u_2};$$

$$C(u_1, u_2) = \left\{1 + \frac{\left[(1+u_1)^{-\theta} - 1\right]\left[(1+u_1)^{-\theta} - 1\right]}{2^{-\theta} - 1}\right\}^{-1/\theta}$$

for $-\infty < \theta < \infty$;

$$C(u_1, u_2) = 1 + \theta\left\{\log\left[\exp\left(\frac{\theta}{u_1 - 1}\right) + \exp\left(\frac{\theta}{u_2 - 1}\right)\right]\right\}^{-1}$$

for $2 \leq \theta < \infty$;

$$C(u_1, u_2) = \theta \left\{ \log \left[\exp\left(\frac{\theta}{u_1}\right) + \exp\left(\frac{\theta}{u_2}\right) - \exp(\theta) \right] \right\}^{-1}$$

for $0 < \theta < \infty$;

$$C(u_1, u_2) = \left\{ \log \left[\exp\left(u_1^{-\theta}\right) + \exp\left(u_2^{-\theta}\right) - \exp(1) \right] \right\}^{-1/\theta}$$

for $0 < \theta < \infty$;

$$C(u_1, u_2) = \left[1 - \left(1 - u_1^{\theta}\right) u_2^{\theta/2} - \left(1 - u_2^{\theta}\right) u_1^{\theta/2} \right]^{1/\theta}$$

for $0 \leq \theta < 1$;

$$C(u_1, u_2) = 1 - \left[(1 - u_1)^{\theta} + (1 - u_2)^{\theta} - (1 - u_1)^{\theta} (1 - u_2)^{\theta} \right]^{1/\theta}$$

for $1 \leq \theta < \infty$;

$$C(u_1, u_2) = \left[u_1^{\theta} u_2^{\theta} - 2 (1 - u_1)^{\theta} (1 - u_2)^{\theta} \right]^{1/\theta}$$

for $0 < \theta \leq 1/2$;

$$C(u_1, u_2) = \left\{ 1 + \left[\left(u_1^{-1} - 1\right)^{\theta} + \left(u_2^{-1} - 1\right)^{\theta} \right]^{1/\theta} \right\}^{-1}$$

for $1 \leq \theta < \infty$;

$$C(u_1, u_2) = \left\{ 1 + \left[\left(u_1^{-1/\theta} - 1\right)^{\theta} + \left(u_2^{-1/\theta} - 1\right)^{\theta} \right]^{1/\theta} \right\}^{\theta}$$

for $0 < \theta \leq 1$;

$$C(u_1, u_2) = \left\{ 1 - \left[\left(1 - u_1^{1/\theta}\right)^{\theta} + \left(1 - u_2^{1/\theta}\right)^{\theta} \right]^{1/\theta} \right\}^{\theta}$$

for $1 \leq \theta < \infty$;

$$C(u_1, u_2) = 1 - \left[1 - \left\{ \left[1 - (1 - u_1)^{\theta}\right]^{1/\theta} + \left[1 - (1 - u_2)^{\theta}\right]^{1/\theta} - 1 \right\}^{\theta} \right]^{1/\theta}$$

for $1 \leq \theta < \infty$.

2.44 Roch and Alegre's copula

Roch and Alegre (2006) have proposed the copula

$$C(u_1, u_2) = \exp \left\{ 1 - \left[\left(((1 - \log u_1)^{\alpha} - 1)^{\delta} + ((1 - \log u_2)^{\alpha} - 1)^{\delta} \right)^{1/\delta} + 1 \right]^{1/\alpha} \right\}$$

for $\alpha > 0$ and $\delta \geq 1$. Independence corresponds to $\alpha = 1$ and $\delta = 1$.

2.45 van der Hoek *et al.*'s copula

For $0 < \beta < 1$, $\beta < \eta < 1$ and a given copula C, let

$$A = [0, \beta] \times [0, \beta],\ B = [\beta, 1] \times [0, \beta],\ C = [0, \beta] \times [\beta, 1],$$
$$E = [\beta, 1] \times [\beta, 1] \cap G,\ F = [\beta, 1] \times [\beta, 1] \cap G^c,$$

where

$$G = \left\{ (u_1, u_2) \in [0, 1]^2 : C \left(\eta + \frac{1-\eta}{1-\beta} (u_1 - \beta), \eta + \frac{1-\eta}{1-\beta} (u_1 - \beta) \right) \leq \eta \right\}.$$

With this notation, van der Hoek *et al.* (2006) have shown that

$$D (u_1, u_2) = \begin{cases} \frac{\beta}{\eta} C \left(\frac{\eta}{\beta} u_1, \frac{\eta}{\beta} u_2 \right), & \text{if } (u_1, u_2) \in A, \\ \frac{\beta}{\eta} C \left(\eta + \frac{1-\eta}{1-\beta} (u_1 - \beta), \frac{\eta}{\beta} u_2 \right), & \text{if } (u_1, u_2) \in B, \\ \frac{\beta}{\eta} C \left(\frac{\eta}{\beta} u_1, \eta + \frac{1-\eta}{1-\beta} (u_2 - \beta) \right), & \text{if } (u_1, u_2) \in C, \\ \frac{\beta}{\eta} C \left(\eta + \frac{1-\eta}{1-\beta} (u_1 - \beta), \eta + \frac{1-\eta}{1-\beta} (u_2 - \beta) \right), & \text{if } (u_1, u_2) \in E, \\ \beta + \frac{1-\beta}{1-\eta} \left[C \left(\frac{\eta}{\beta} u_1, \eta + \frac{1-\eta}{1-\beta} (u_2 - \beta) \right) - \eta \right], & \text{if } (u_1, u_2) \in F \end{cases}$$

is a valid copula. van der Hoek *et al.* (2006) refer to this copula as the single kink copula.

2.46 Durante *et al.*'s copula

Let $f : [0, 1] \rightarrow [0, 1]$ and let $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(p)}$ denote sorted values of $u_1, u_2, \ldots, u_p$. Durante *et al.* (2007b) have shown that

$$C (u_1, \ldots, u_p) = u_{(1)} \prod_{i=2}^{p} f \left(u_{(i)} \right)$$

is a valid copula if and only if $f(1) = 1$, f is increasing and $f(t)/t$ is decreasing on $(0, 1]$. In particular,

$$C (u_1, u_2) = \min (u_1, u_2) \prod_{i=2}^{p} f (\max (u_1, u_2))$$

is a valid copula if and only if $f(1) = 1$, f is increasing and $f(t)/t$ is decreasing on $(0, 1]$, see Durante (2006, 2007).

2.47 Klement *et al.*'s copula

Let $0 < b < 1$ and let $h : [0,1] \to [0,1]$ be a 1-Lipschitz function such that $\max(w + b - 1, 0) \leq h(w) \leq \min(w, b)$ for all $0 \leq w \leq 1$. Klement *et al.* (2007) have shown that

$$C(u_1, u_2) = \begin{cases} \dfrac{h(u_1)\, u_2}{b}, & \text{if } u_2 \leq b, \\ \\ \dfrac{(1 - u_2)\, h(u_1) + (u_2 - b)\, u_1}{1 - b}, & \text{if } u_2 > b, \end{cases}$$

$$C(u_1, u_2) = \begin{cases} u_2, & \text{if } u_2 \leq h(u_1), \\ \\ h(u_1), & \text{if } h(u_1) < u_2 \leq b, \\ \\ u_2 - b + h(u_1), & \text{if } b < u_2 \leq u_1 + b - h(u_1), \\ \\ u_1, & \text{otherwise} \end{cases}$$

and

$$C(u_1, u_2) = \begin{cases} 0, & \text{if } u_2 \leq b - h(u_1), \\ \\ u_2 - b + h(u_1), & \text{if } b - h(u_1) < u_2 \leq b, \\ \\ h(u_1), & \text{if } b < u_2 \leq 1 - u_1 + h(u_1), \\ \\ u_1 + u_2 - 1, & \text{otherwise} \end{cases}$$

are valid copulas.

2.48 Durante and Jaworski's copulas

Let $\delta : [0,1] \to [0,1]$ be such that i) $\max(2w - 1, 0) \leq \delta(w) \leq 1$ for all $0 \leq w \leq 1$; ii) $\delta(v) \leq \delta(w)$ for every $v \leq w$; iii) $\mid \delta(v) - \delta(w) \mid \leq 2 \mid v - w \mid$ for all $0 \leq v, w \leq 1$. iv) $\delta^{'}(w)$ exists; v) there exists $e > 0$ such that the supremum of $\delta^{'}(w)$ is equal to $2/(1 + 2e)$. Under these conditions, Durante and Jaworski (2008) have shown that

$$C(u_1, u_2) = \min\left[u_1, u_2, \lambda\delta(u_1) + (1 - \lambda)\delta(u_2)\right]$$

and

$$C(u_1, u_2) = \frac{1}{2e} \int_{1/2-e}^{1/2+e} \min\left[u_1, u_2, \lambda\delta(u_1) + (1 - \lambda)\delta(u_2)\right] d\lambda$$

are valid copulas for all $1/2 - e \leq \lambda \leq 1/2 + e$.

Durante and Jaworski (2008) have also provided two further constructions. Assume now that $\delta^{+}, \delta^{-} : [0,1] \to [0,1]$ are functions satisfying conditions (i)-(iii) such that $\delta(w) = \left\{\delta_{+}(w)\delta_{-}(w)\right\}/2$

and $\delta_-(w) \leq \delta(w) \leq \delta_+(w)$ for all $w_\ell < w < w_r$, where $w_\ell = \sup\{w \in [0,1] : \delta(w) = 0\}$ and $w_r = \sup\{w \in [0,1] : \delta(w) = 2w - 1\}$. Under these further conditions, Durante and Jaworski (2008) have also shown that

$$C(u_1, u_2) = \min\left[u_1, u_2, \frac{\lambda\left[\delta_+(u_1) + \delta_+(u_2)\right] + (1-\lambda)\left[\delta_-(u_1) + \delta_-(u_2)\right]}{2}\right]$$

and

$$C(u_1, u_2) = \int_0^1 \min\left[u_1, u_2, \frac{\lambda\left[\delta_+(u_1) + \delta_+(u_2)\right] + (1-\lambda)\left[\delta_-(u_1) + \delta_-(u_2)\right]}{2}\right] d\lambda$$

are valid copulas for all $0 \leq \lambda \leq 1$.

2.49 Semilinear copulas

Let $\delta : [0,1] \to [0,1]$ be such that $\delta(0) = 0$, $\delta(1) = 1$, $\delta(v) \leq v$ for all $0 \leq v \leq 1$, δ is increasing and $0 \leq \delta(v) - \delta(w) \leq 2(v - w)$ for all $0 \leq v < w \leq 1$. Durante *et al.* (2008) have shown that

$$C(u_1, u_2) = \begin{cases} u_2 \dfrac{\delta(u_1)}{u_1}, & \text{if } u_2 \leq u_1, \\ \\ u_1 \dfrac{\delta(u_2)}{u_2}, & \text{if } u_2 > u_1 \end{cases} \tag{6}$$

is a valid copula if and only if $(\delta(u)/u)^{'} \geq 0$ and $\left(\delta(u)/u^2\right)^{'} \leq 0$ for all $0 \leq u \leq 1$, where the derivatives are assumed to exist. Copulas taking the form (6) are referred to as semilinear copulas.

de Baets *et al.* (2007) have shown that

$$C(u_1, u_2) = \begin{cases} u_2 \dfrac{\delta(u_1)}{u_1}, & \text{if } u_2 \leq u_1, \\ \\ \dfrac{u_1(u_2 - u_1)}{1 - u_1} + \dfrac{1 - u_2}{1 - u_1}\delta(u_1), & \text{if } u_2 > u_1 \end{cases} \tag{7}$$

is a valid copula if and only if $\delta(u)/u$ is increasing, $\{u - \delta(u)\}/(1-u)$ is increasing, and $\delta(u) \geq u^2$ for all $0 \leq u \leq 1$. Copulas taking the form (7) are referred to as asymmetric semilinear copulas.

Let $\omega : [0,1] \to [0,1/2]$ be such that $\delta(v) \leq \min(v, 1-v)$ for all $0 \leq v \leq 1$, and $\mid \omega(v) - \omega(w) \mid \leq \mid v - w \mid$ for all $0 \leq v < w \leq 1$. de Baets *et al.* (2007) have shown further that

$$C(u_1, u_2) = \begin{cases} \dfrac{u_2}{1 - u_1}\omega(u_1), & \text{if } u_1 + u_2 \leq 1, \\ \\ u_1 + u_2 - 1 + \dfrac{1 - u_2}{u_1}\omega(u_1), & \text{if } u_1 + u_2 > 1 \end{cases} \tag{8}$$

is a valid copula if and only if $\omega(u)/u$ is decreasing, and $\{u - \omega(u)\}/u^2$ is decreasing. Copulas taking the form (8) are referred to as vertical semilinear copulas.

Supposing $\delta(1/2) = \omega(1/2)$, de Baets *et al.* (2007) have shown further that

$$C\left(u_1, u_2\right) = \begin{cases} \dfrac{u_1 + u_2 - 1}{2u_2 - 1}\delta\left(u_2\right) + \dfrac{u_2 - u_1}{2u_2 - 1}\omega\left(1 - u_2\right), \\ \qquad \text{if either } u_1 \leq u_2,\ u_1 + u_2 > 1 \text{ or } u_1 > u_2,\ u_1 + u_2 \leq 1, \\ \\ \dfrac{u_1 + u_2 - 1}{2u_1 - 1}\delta\left(u_1\right) + \dfrac{u_1 - u_2}{2u_1 - 1}\omega\left(u_1\right), \\ \qquad \text{otherwise} \end{cases} \tag{9}$$

is a valid copula if and only if

$$\frac{\omega(u) - \delta(u)}{1 - 2u}, \ \frac{\omega(1-u) - \delta(u)}{1 - 2u}$$

are increasing in $[0, 1/2) \cup (1/2, 1]$,

$$\omega(u) + \omega(1-u) \leq \frac{(1-2u)\delta(u) - (1-2v)\delta(u)}{v - u}, \ \delta(u) + \delta(1-u) \geq \frac{(1-2u)\omega(v) - (1-2v)\omega(u)}{v - u}$$

for all $0 \leq u < v < 1/2$, and

$$\omega(v) + \omega(1-v) \leq \frac{(1-2u)\delta(u) - (1-2v)\delta(u)}{v - u}, \ \delta(v) + \delta(1-v) \geq \frac{(1-2u)\omega(v) - (1-2v)\omega(u)}{v - u}$$

for all $1/2 < u < v \leq 1$. Copulas taking the form (9) are referred to as orbital semilinear copulas.

Let $u, v : [0, 1] \to \mathbb{R}$ be absolutely continuous functions satisfying

$$\lim_{u_1 \to 0, 0 \leq u_2 \leq u_1} u_2\left(u_1 - u_2\right) u\left(u_1\right) = 0, \qquad \lim_{u_2 \to 0, 0 \leq u_1 \leq u_2} u_1\left(u_2 - u_1\right) v\left(u_2\right) = 0.$$

Jwaid *et al.* (2013) have shown that

$$C\left(u_1, u_2\right) = \begin{cases} \dfrac{u_1}{u_2}\delta\left(u_2\right) - u_1\left(u_2 - u_1\right) v\left(u_2\right), & \text{if } 0 < u_1 \leq u_2, \\ \\ \dfrac{u_2}{u_1}\delta\left(u_1\right) - u_2\left(u_1 - u_2\right) u\left(u_1\right), & \text{if } 0 < u_2 \leq u_1 \end{cases} \tag{10}$$

is a valid copula if and only if $u(1) = v(1) = 0$ and

$$\max\left[u(w) + w\left|u^{'}(w)\right|, v(w) + w\left|v^{'}(w)\right|\right] \leq \left(\frac{\delta(w)}{w}\right)^{'}, \ u(w) + v(w) \geq w\left(\frac{\delta(w)}{w^2}\right)^{'}$$

for all $0 \leq w \leq 1$, where the derivatives are assumed to exist. Copulas taking the form (10) are referred to as semiquadratic copulas.

2.50 Kolesárová *et al.*'s copula

Given two two-dimensional copulas, C_1 and C_2, Kolesárová *et al.* (2008) have shown that

$$C\left(u_1, u_2, u_3\right) = \int_0^{u_2} \frac{\partial}{\partial w} C_1\left(u_1, w\right) \frac{\partial}{\partial w} C_2\left(w, u_3\right) dw$$

is a valid three-dimensional copula. A p-variate generalization is: if $C_1, \ldots, C_p$ are two-dimensional copulas then

$$C\left(u_1, u_2, \ldots, u_p\right) = \int_0^1 \frac{\partial}{\partial w} C_1\left(u_1, w\right) \frac{\partial}{\partial w} C_2\left(u_2, w\right) \cdots \frac{\partial}{\partial w} C_p\left(u_p, w\right) dw$$

is a valid p-dimensional copula.

2.51 Liebscher's copula

Let Ψ and ψ_{jk} be functions from $[0,1]$ to $[0,1]$ such that i) Ψ is absolutely monotonic of order p on $[0,1]$, and $\Psi(0)=0$; ii) ψ_{jk} are differentiable and monotonically increasing with $\psi_{jk}(0)=0$ and $\psi_{jk}(1)=1$ for all k, j; iii) the condition

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m}\psi_{jk}(v)\right)=v$$

holds for all $0<v<1$ and for all $k=1,\ldots,p$. Under these conditions, Liebscher (2008) has shown that

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m}\psi_{j1}\left(u_1\right)\cdots\psi_{jp}\left(u_p\right)\right)$$

is a valid copula. Under similar conditions and given arbitrary copulas, C_j, $j=1,\ldots,m$, Fischer and Köck (2012) have shown that

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m}C_j\left(\psi_{j1}\left(u_1\right),\ldots,\psi_{jp}\left(u_p\right)\right)\right)$$

is also a valid copula.

2.52 Nelson *et al.*'s copula

Let $T_U=\{(u_1,u_2):0\leq u_1\leq u_2\leq 1\}$, $T_L=\{(u_1,u_2):0\leq u_2\leq u_1\leq 1\}$ and $D=T_U\cap T_L$. Given two copulas, C_1 and C_2, their *diagonal splice* (Definition 3, Nelson *et al.*, 2008) is defined by

$$\left(C_1\Box C_2\right)\left(u_1,u_2\right)=\begin{cases}C_1\left(u_1,u_2\right), & \text{if } (u_1,u_2)\in T_U,\\ \\ C_2\left(u_1,u_2\right), & \text{if } (u_1,u_2)\in T_L\backslash D.\end{cases}$$

Nelson *et al.* (2008) have shown that $C_1\Box C_2$ is itself a copula if and only if

$$\int_0^{u_1}\frac{\partial^2 C_1\left(u_1,u_2\right)}{\partial u_1\partial u_2}du_2=\int_0^{u_1}\frac{\partial^2 C_2\left(u_1,u_2\right)}{\partial u_1\partial u_2}du_2$$

for all $0\leq u_1\leq 1$.

2.53 Quesada-Molina *et al.*'s copula

Let $\delta_{x_0}:[0,1-x_0]\rightarrow[0,1-x_0]$ denote a function satisfying i) $\delta_{x_0}\left(1-x_0\right)=1-x_0$, ii) $0\leq\delta_{x_0}(u)\leq u$ for all $0\leq u\leq 1-x_0$, and iii) $0\leq\delta_{x_0}(u)-\delta_{x_0}(v)\leq 2(u-v)$ for all $0\leq u\leq v\leq 1-x_0$.

Let

$$
\begin{aligned}
&S_L(x_0) = [0, x_0]^2, \ S_U(x_0) = [1 - x_0, 1]^2, \ D(x_0) = S_L(x_0) \cup S_U(x_0), \\
&m_{x_0}(u, v) = \max\left[\min(u - x_0, v), 0\right], \\
&M_{x_0}(u, v) = \min\left[\max(u - x_0, v), 1 - x_0\right], \\
&k_{x_0}(u, v) = \frac{1}{2}\delta_{x_0}\left(m_{x_0}(u, v)\right) + \frac{1}{2}\delta_{x_0}\left(M_{x_0}(u, v)\right), \\
&k^*_{x_0}(u, v) = k_{x_0}\left(\max(u, v), \min(u, v)\right), \\
&A_{x_0}(u, v) = u - 1 + x_0 + k_{x_0}(v, 1 - x_0).
\end{aligned}
$$

With this notation, Quesada-Molina *et al.* (2008) have shown that

$$
C(u_1, u_2) = \begin{cases}
\min\left[u_1, u_2, k^*_{x_0}(u_1, u_2)\right], & \text{if } (u_1, u_2) \in [0,1]^2 \backslash D(x_0), \\
\\
\min\left[k^*_{x_0}(x_0, u_1), k^*_{x_0}(x_0, u_2)\right], & \text{if } (u_1, u_2) \in S_L(x_0), \\
\\
\min\left[A_{x_0}(u_1, u_2), A_{x_0}(u_2, u_1)\right], & \text{if } (u_1, u_2) \in S_U(x_0)
\end{cases}
$$

is a valid copula.

2.54 Gluing of copulas

Suppose C_1 and C_2 are valid copulas and let $0 \leq \theta \leq 1$. Then, Siburg and Stoimenov (2008) have shown that

$$
C(u_1, \ldots, u_p) = \begin{cases}
\theta C_1\left(u_1, \ldots, \dfrac{u_i}{\theta}, \ldots, u_p\right), & \text{if } 0 \leq u_i \leq \theta, \\
\\
\theta C_1(u_1, \ldots, 1, \ldots, u_p) + (1 - \theta) C_2\left(u_1, \ldots, \dfrac{u_i - \theta}{1 - \theta}, \ldots, u_p\right), & \text{if } \theta < u_i \leq 1
\end{cases}
$$

is a valid copula. In the case $p = 2$, for example,

$$
C(u_1, u_2) = \begin{cases}
\theta C_1\left(\dfrac{u_1}{\theta}, u_2\right), & \text{if } 0 \leq u_1 \leq \theta, \\
\\
\theta + (1 - \theta) C_2\left(\dfrac{u_1 - \theta}{1 - \theta}, u_2\right), & \text{if } \theta < u_1 \leq 1
\end{cases}
$$

and

$$
C(u_1, u_2) = \begin{cases}
\theta C_1\left(u_1, \dfrac{u_2}{\theta}\right), & \text{if } 0 \leq u_2 \leq \theta, \\
\\
\theta + (1 - \theta) C_2\left(u_1, \dfrac{u_2 - \theta}{1 - \theta}\right), & \text{if } \theta < u_2 \leq 1
\end{cases}
$$

are valid copulas, as shown by Mesiar *et al.* (2010).

In case of more than two given copulas, say $C_1, \ldots, C_N$, and given $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$, Siburg and Stoimenov (2008) have shown that C defined by

$$
C(u_1, \ldots, u_p) = (\theta_k - \theta_{k-1}) C_k\left(u_1, \ldots, \frac{u_i - \theta_{k-1}}{\theta_k - \theta_{k-1}}, \ldots, u_p\right)
$$

for $\theta_{k-1} \leq u_i \leq \theta_k$, $k = 1, \ldots, N$ is a valid copula. These operations are referred to as gluing.

2.55 Úbeda-Flores's copula

For given $f, g, h, j : [0,1]^{p-1} \to [0,1]$, Úbeda-Flores (2008) has shown that

$C(u_1, \ldots, u_p) = f(u_1, \ldots, u_{p-1}) u_p^3 + g(u_1, \ldots, u_{p-1}) u_p^2 + h(u_1, \ldots, u_{p-1}) u_p + j(u_1, \ldots, u_{p-1})$

is a valid copula under suitable conditions (Theorem 1, Úbeda-Flores, 2008).

2.56 Amblard and Girard's copula

Amblard and Girard (2009) have shown that

$$C(u_1, u_2) = u_1 u_2 + \theta(\min(u_1, u_2)) \phi(u_1) \phi(u_2)$$

is a copula, where $\theta : [0,1] \to \mathbb{R}$ and $\phi : [0,1] \to \mathbb{R}$ are such that $\phi(0) = 0$, $\theta(1)\phi(1) = 0$, $\phi^{'}(u_1)(\theta\phi)^{'}(u_2) \geq 1$ for all $0 < u_1 \leq u_2 < 1$ and $\theta^{'}(u_1) \leq 0$ for all $0 \leq u_1 \leq 1$.

2.57 Fourier copulas

Ibragimov (2009) has defined Fourier copulas as

$$C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} [1 + g(x, y)] \, dxdy,$$

where

$$g(x, y) = \sum_{j=1}^{N} \left[\alpha_j \sin\left(2\pi \left(\beta_1^j x + \beta_2^j y\right)\right) + \gamma_j \cos\left(2\pi \left(\beta_1^j x + \beta_2^j y\right)\right) \right]$$

for $N \geq 1$, $-\infty < \alpha_j, \gamma_j < \infty$ and $\beta_1^j, \beta_2^j \in \{\ldots, -1, 0, 1, \ldots\}$ arbitrary numbers such that $\beta_1^j + \beta_2^j \neq 0$ for all $j_1, j_2 \in \{1, \ldots, N\}$ and

$$1 + \sum_{j=1}^{N} (\alpha_j \delta_j + \gamma_j \delta_{j+N}) \geq 0$$

for $-1 \leq \delta_1, \ldots, \delta_{2N} \leq 1$.

2.58 Ahmadi-Clayton copula

Let $\theta_i \geq 0$, $i = 1, \ldots, p$ be such that $\theta_i \geq \theta_{i-1}$, $i = 2, \ldots, p$. Let n_i, $i = 1, \ldots, p$ be integers summing to n. Let Π denote a permutation matrix with each row and each column contains only one element equal to one and the remaining elements equal to zero. Javid (2009) has defined the following extension of the Clayton copula

$$C(u_1, \ldots, u_n) = \prod_{i=1}^{p} \left[\sum_{j=n_{i-1}+1}^{n_i} u_j^{-\theta_i} - n_i + n_{i-1} + 1 \right]^{-1/\theta_i},$$

where $(z_1, \ldots, z_n)^T = \Pi (u_1, \ldots, u_n)^T$.

2.59 Lévy-frailty copulas

Let $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(p)}$ denote sorted values of $u_1, u_2, \ldots, u_p$ and let a_i, $i = 0, 1, \ldots, p-1$ denote some real numbers. Mai and Scherer (2009) have shown that

$$C\left(u_1, \ldots, u_p\right) = \prod_{i=1}^{p} u_{(i)}^{a_{i-1}} \tag{11}$$

is a valid copula if and only if $a_0 = 1$ and $\{a_i\}$ are p-monotone; that is, $\Delta^{j-1} a_k = 0$ for all $k = 0, 1, \ldots, p-1$ and $j = 1, 2, \ldots, p-k$, where

$$\Delta^j a_k = \sum_{i=0}^{j} (-1)^i \binom{j}{i} a_{k+i}$$

for $j \geq 0$ and $k \geq 0$. Copulas defined by (11) are referred to as Lévy-frailty copulas.

2.60 Rodriguez-Lallena's copula

Let $a : [0,1] \to [0,1]$ and $b : [0,1] \to (0,\infty)$ be given functions satisfying the following conditions: i) $\max\left(a\left(y\right) + y - 1, 0\right) \leq b\left(y\right) \leq \min\left(a\left(y\right), y\right)$ for all $0 \leq y \leq 1$; ii) there exists a partition $0 = y_0 < y_1 < y_2 < \cdots < y_n = 1$ of $[0,1]$ such that, for each $i = 1, \ldots, n$, a and b are differentiable on (y_{i-1}, y_i) and either $a^{'}(y) < 0$, $a^{'}(y) = 0$ or $a^{'}(y) > 0$ for every $y_{i-1} < y < y_i$; iii) $a^{'}(y)\left[b(y) - ya(y)\right] \geq 0$ for all $0 < y < 1$ such that $a^{'}(y)$ exists; iv) $a^{'}(y)b(y)/a(y) \leq b^{'}(y)$ whenever $a(y) \neq 0$ and both $a^{'}(y)$ and $b^{'}(y)$ exist; v) $b^{'}(y) \leq 1 + a^{'}(y)\left[y - b(y)\right] / \left[1 - a(y)\right]$ whenever $a(y) \neq 1$ and both $a^{'}(y)$ and $b^{'}(y)$ exist. Under these conditions, Rodriguez-Lallena (2009) has shown that

$$C\left(u_1, u_2\right) = \begin{cases} u_1 \dfrac{b\left(u_2\right)}{a\left(u_2\right)}, & \text{if } 0 \leq u_1 < a\left(u_2\right), \\ b\left(u_2\right), & \text{if } u_1 = a\left(u_2\right), \\ u_2 - \left(1 - u_1\right) \dfrac{u_2 - b\left(u_2\right)}{1 - a\left(u_2\right)}, & \text{if } a\left(u_2\right) < u_1 \leq 1 \end{cases}$$

is a valid copula.

2.61 Yang *et al.*'s copula

Let U_i, $i = 1, \ldots, n$ be uniform $[0,1]$ random variables. Suppose there exists a uniform $[0,1]$ random variable U such that U_i, $i = 1, \ldots, n$ are independent conditionally on U. Suppose also that U_i and U have the joint cumulative distribution function

$$a_{i,1} \min\left(u_i, u\right) + a_{i,3} u_i u + a_{i,2} \max\left(u_i + u - 1, 0\right)$$

for $0 \leq a_{i,1}, a_{i,2}, a_{i,3} \leq 1$ and $a_{i,1} + a_{i,2} + a_{i,3} = 1$. For $(j_1, \ldots, j_n)$, where $j_i \in \{1, 2, 3\}$, write

$$C^{(j_1, \ldots, j_n)}\left(u_1, \ldots, u_n\right) = \max\left[\min_{1 \leq i \leq n, j_i = 1} u_i + \min_{1 \leq i \leq n, j_i = 3} u_i - 1, 0\right] \prod_{1 \leq i \leq n, j_i = 2} u_i.$$

Yang *et al.* (2009) have shown that the joint distribution of $(U_1, \ldots, U_n)$ can be expressed as

$$C(u_1, \ldots, u_n) = \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) C^{(j_1,\ldots,j_n)}(u_1, \ldots, u_n),$$

which is a copula.

2.62 Zhang's copula

Zhang (2009) has proposed a copula defined by

$$C(u_1, \ldots, u_p) = \prod_{j=1}^{p} \min_{1 \leq d \leq D} \left(u_d^{a_{j,d}} \right)$$

for $a_{j,d} \geq 0$ and $a_{1,d} + \cdots + a_{p,d} = 1$ for all $d = 1, \ldots, D$.

2.63 Andronov's copula

Let $x = q(u)$ denote the root of the equation

$$u = 1 - \frac{1}{p} \sum_{j=0}^{p-1} \frac{p-j}{j!} x^j \exp(-x).$$

Andronov (2010) has shown that

$$\begin{aligned} C(u_1, \ldots, u_p) &= 1 - \sum_{j=0}^{p-1} \frac{1}{j!} q^j(u_1) \exp[-q(u_1)] + \sum_{i=0}^{p} \frac{q(u_1) \cdots q(u_{i-1})}{p(p-1) \cdots (p-i+2)} \\ & \cdot \sum_{j=0}^{p-i} \frac{1}{j!} \left\{ q^j(u_{i-1}) \exp[-q(u_{i-1})] - q^j(u_i) \exp[-q(u_i)] \right\} \end{aligned}$$

is a valid copula.

2.64 Durante and Salvadori's copulas

Let $F_i : [0,1] \to [0,1]$, $i = 1, \ldots, p$ be continuous and strictly increasing functions such that $F_i(0) = 0$, $F_i(1) = 1$, $G_i(u) = u / F_i(u)$ is strictly increasing and $G_i(0^+) = 0$. For given copulas, A and B, Durante and Salvadori (2010) have shown that

$$C(u_1, \ldots, u_p) = A(F_1(u_1), \ldots, F_p(u_p)) B(G_1(u_1), \ldots, G_p(u_p))$$

are valid copulas.

Furthermore, let $0 \leq \lambda_{ij} \leq 1$, $\lambda_{ij} = \lambda_{ji}$ and $\sum_{j=1, j \neq i}^{p} \lambda_{ij} \leq 1$. Durante and Salvadori (2010) have also shown that

$$C(u_1, \ldots, u_p) = \left(\prod_{i=1}^{p} u_i \right)^{1 - \sum_{j=1, j \neq i}^{p} \lambda_{ij}} \prod_{i<j} [\min(u_i, u_j)]^{\lambda_{ij}}$$

are valid copulas.

2.65 Cube copula

Holman and Ritter (2010) have defined what is referred to as a Cube copula as

$$C(u_1, u_2) = \begin{cases} q_2 u_1 u_2, & \text{if } u_1 \leq a, u_2 \leq a, \\ u_1 \left[q_2 a + q_1 (u_2 - a) \right], & \text{if } u_1 \leq a < u_2, \\ u_2 \left[q_2 a + q_1 (u_1 - a) \right], & \text{if } u_2 \leq a < u_1, \\ q_2 a^2 + q_1 a (u_1 + u_2 - 2a) + q_0 (u_1 - a)(u_2 - a), & \text{if } u_1 > a, u_2 > a \end{cases}$$

for some suitable constants q_0, q_1, q_2 and a.

2.66 Komelj and Perman's copulas

Let $g_.[0,1] \rightarrow \mathbb{R}$, $i = 1, \ldots, p$ be continuous functions not identically equal to zero. Let $\int_0^1 g_i(t)dt = 0$, $G_i(t) = \int_0^t g_i(u)du$ and

$$\begin{aligned} & a_i = -\min_{0 \leq t \leq 1} g_i(t), \; b_i = \max_{0 \leq t \leq 1} g_i(t), \\ & M_{ij}^a = \max(a_i a_j, b_i b_j), \; M_{ij}^b = \max(a_i b_j, b_i a_j), \; M_{ij} = \max\left(M_{ij}^a, M_{ij}^b\right), \\ & M^a = \sum_{1 \leq i < j \leq p} M_{ij}^a, \; M^b = \sum_{1 \leq i < j \leq p} M_{ij}^b, \; M = \sum_{1 \leq i < j \leq p} M_{ij}. \end{aligned}$$

Let θ_{ij}, $1 \leq i < j \leq p$ be constants taking values in $[-1/M, 1/M]$ or $[-1/M^a, 0]$ or $\left[0, 1/M^b\right]$. With these assumptions, Komelj and Perman (2010) have shown that

$$C(u_1, \ldots, u_p) = \prod_{i=1}^p u_i + \sum_{1 \leq i < j \leq p} \theta_{ij} G_i(u_i) G_j(u_j) \prod_{k=1, k \neq i, k \neq j}^p u_k$$

are valid copulas.

2.67 DUCS copula

Let $f : [0,1] \rightarrow [0, \infty]$ be a continuous strictly decreasing convex function satisfying $f(1) = 0$ and let $d : [0,1] \rightarrow [0,1]$. Mesiar and Pekárová (2010) have shown that

$$C(u_1, u_2) = u_1 f^{-1}\left(\frac{f(u_2)}{d(u_1)}\right)$$

is a valid copula referred to as the DUCS copula if and only if there exists a function $e : [0,1] \rightarrow [0,1]$ such that $d(u)e(u) = u$ for all $0 \leq u \leq 1$, and both d and e are non-decreasing on $(0,1]$.

2.68 de Baets *et al.*'s copulas

Let A denote a copula and let $R = [a_1, a_2] \times [b_1, b_2]$ be such that $a_1 < a_2$, $b_1 < b_2$ and $A(R) > 0$. For any given copula D, de Baets *et al.* (2011) have shown that

$$C(u_1, u_2) = \begin{cases} A(R)\theta D\left(\dfrac{A([a_1, u_1] \times [b_1, b_2])}{A(R)}, \dfrac{A([a_1, a_2] \times [b_1, u_2])}{A(R)}\right) \\ \quad + A(u_1, b_1) + A(a_1, u_2) - A(a_1, b_1), & \text{if } (u_1, u_2) \in R, \\ \\ A(u_1, u_2), & \text{if } (u_1, u_2) \in [0,1]^2 \setminus R \end{cases}$$

are valid copulas.

Moreover, let A denote a copula such that $0 < \mu = A(1/2, 1/2) < 1/2$, and let $R_{11} = [0, 1/2]^2$, $R_{12} = [0, 1/2] \times [1/2, 1]$, $R_{21} = [1/2, 1] \times [0, 1/2]$, $R_{22} = [1/2, 1]^2$, $u(x) = A(x, 1/2)$ and $v(y) = A(1/2, y)$. For any given copulas, C_{11}, C_{12}, C_{21} and C_{22}, de Baets *et al.* (2011) have also shown that

$$C(u_1, u_2) = \begin{cases} \mu C_{11}\left(\dfrac{u(u_1)}{\mu}, \dfrac{v(u_2)}{\mu}\right), \\ \quad \text{if } (u_1, u_2) \in R_{11}, \\ \\ \left(\dfrac{1}{2} - \mu\right) C_{12}\left(\dfrac{u_1 - u(u_1)}{1/2 - \mu}, \dfrac{v(u_2) - \mu}{1/2 - \mu}\right) + u(u_1), \\ \quad \text{if } (u_1, u_2) \in R_{12}, \\ \\ \left(\dfrac{1}{2} - \mu\right) C_{21}\left(\dfrac{u(u_1) - \mu}{1/2 - \mu}, \dfrac{u_2 - v(u_2)}{1/2 - \mu}\right) + v(u_2), \\ \quad \text{if } (u_1, u_2) \in R_{21}, \\ \\ \mu C_{22}\left(\dfrac{u_1 + \mu - 1/2 - u(u_1)}{\mu}, \dfrac{u_2 + \mu - 1/2 - v(u_2)}{\mu}\right) + u(u_1) + v(u_2) - \mu, \\ \quad \text{if } (u_1, u_2) \in R_{22} \end{cases}$$

are valid copulas.

2.69 Biconic copula

Let $\delta : [0, 1] \to [0, 1]$ be such that it is convex, $\delta(u)$ is increasing, 2-Lipschitz and $\max(2u - 1, 0) \leq \delta(u) \leq 1$ for all $0 \leq u \leq 1$. Under these assumptions, Durante and Fernández-Sánchez (2011) have shown that

$$C(u_1, u_2) = \begin{cases} (u_2 + 1 - u_1)\, \delta\left(\dfrac{u_2}{u_2 + 1 - u_1}\right), & \text{if } u_1 \geq u_2, \\ \\ (u_1 + 1 - u_2)\, \delta\left(\dfrac{u_1}{u_1 + 1 - u_2}\right), & \text{if } u_1 < u_2 \end{cases}$$

is a valid copula referred to as a Biconic copula.

2.70 Klein *et al.*'s copula

For $r \geq 1$, $0 < \alpha < 1$ and for given copulas, C_1 and C_2, such that

$$\left[\frac{\partial \log C_1 \left(u_1, u_2 \right)}{\partial u_1} - \frac{\partial \log C_2 \left(u_1, u_2 \right)}{\partial u_1} \right] \left[\frac{\partial \log C_1 \left(u_1, u_2 \right)}{\partial u_2} - \frac{\partial \log C_2 \left(u_1, u_2 \right)}{\partial u_2} \right] \geq 0,$$

Klein *et al.* (2011) have shown that

$$C \left(u_1, u_2 \right) = \left[\alpha C_1^r \left(u_1, u_2 \right) + (1 - \alpha) C_1^r \left(u_1, u_2 \right) \right]^{\frac{1}{r}}$$

is a valid copula referred to as a weighted power mean copula.

2.71 Ordinal sum copulas

Let J denote a finite or countable subset of natural numbers. Let C_k, $k \in J$ be given copulas and let (a_k, b_k) be intervals for $k \in J$. Under suitable conditions, Sempi (2011) has shown that

$$C \left(u_1, \ldots, u_p \right) = \begin{cases} a_k + \left(b_k - a_k \right) C_k \left(\dfrac{\min \left(u_1, b_k \right) - a_k}{b_k - a_k}, \ldots, \dfrac{\min \left(u_p, b_k \right) - a_k}{b_k - a_k} \right), \\ \qquad \text{if } a_k < \min \left(u_1, \ldots, u_p \right) < b_k, \\ \\ \min \left(u_1, \ldots, u_p \right), \\ \qquad \text{otherwise} \end{cases}$$

are valid copulas for $k \in J$.

Let C be a copula and let $0 < x_0 < 1$ be such that $C \left(x_0, 1 - x_0 \right) = 0$. Under suitable conditions, Sempi (2011) has also shown that there are copulas, C_1 and C_2, such that

$$C \left(u_1, u_2 \right) = \begin{cases} x_0 C_1 \left(\dfrac{u_1}{x_0}, \dfrac{x_0 + u_2 - 1}{x_0} \right), & \text{if } \left(u_1, u_2 \right) \in [0, x_0] \times [1 - x_0, 1], \\ \\ \left(1 - x_0 \right) C_2 \left(\dfrac{u_1 - x_0}{1 - x_0}, \dfrac{u_2}{1 - x_0} \right), & \text{if } \left(u_1, u_2 \right) \in [x_0, 1] \times [0, 1 - x_0], \\ \\ \max \left(0, u_1 + u_2 - 1 \right), & \text{otherwise} \end{cases}$$

are valid copulas.

2.72 Generalized beta copula

Let $\gamma(a, x)$ and $I_x(a, b)$ denote the incomplete gamma function ratio and the incomplete beta function ratio defined by

$$\gamma(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} \exp(-t) dt$$

and

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

respectively, where

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt$$

and

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

are the gamma and beta functions, respectively. Let $\gamma^{-1}(a,x)$ and $I_x^{-1}(a,b)$ denote the inverse functions of $\gamma(a,x)$ and $I_x(a,b)$ with respect to x. With this notation, Yang *et al.* (2011) have defined the generalized beta copula as

$$C(u_1,\ldots,u_p) = \int_0^\infty \prod_{i=1}^p \gamma^{-1}\left(\frac{I_{u_i}^{-1}(p_i,q)}{\theta - \theta I_{u_i}^{-1}(p_i,q)}, p_i\right) \frac{\theta^{-q-1}\exp(-1/\theta)}{\Gamma(q)} d\theta.$$

The particular case for $p_i = 1$ for all i is

$$\begin{aligned} C(u_1,\ldots,u_p) &= \left[\sum_{i=1}^p u_i - p + 1\right] + \sum_{i_1<i_2}\left[(1-u_{i_1})^{-\frac{1}{q}} + (1-u_{i_2})^{-\frac{1}{q}} - 1\right]^{-q} + \\ &\quad \cdots + (-1)^p\left[\sum_{i=1}^p (1-u_i)^{-\frac{1}{q}} - p + 1\right]^{-q}, \end{aligned}$$

a copula due to Al-Hussaini and Ateya (2006).

2.73 de Meyer *et al.*'s copula

For $\delta : [0,1] \to \mathbb{R}$ a twice differentiable function, $\lambda(u) = \delta(u)/u$, and $\overline{\lambda}(u) = \{u - \delta(u)\}/(1-u)$, de Meyer *et al.* (2012) have shown that

$$C(u_1,u_2) = \begin{cases} u_1 \lambda\left(\dfrac{u_2 + (a-1)u_1}{a}\right), & \text{if } u_1 \leq u_2, u_2 + (a-1)u_1 \leq 1, \\ u_1 - (1-u_2)\overline{\lambda}\left(\dfrac{u_1 + (b-1)u_2}{b}\right), & \text{if } u_1 \leq u_2, (b-1)(1-u_2) \leq u_1 \end{cases}$$

is a valid copula under suitable conditions. This is referred to as a variolinear copula.

2.74 Dolati's copulas

Dolati (2012) has proposed three different copulas. The first is given by

$$C(u_1,\ldots,u_p) = \prod_{i=1}^p u_i + \theta \prod_{i=1}^p f_i(u_i),$$

where $f_i : [0,1] \to \mathbb{R}$, $i = 1, \ldots, p$ are non-zero absolutely continuous functions such that $f_i(0) = f_i(1) = 0$ and

$$-\frac{1}{\sup\limits_{(u_1,\ldots,u_p)\in D^+} \left[\prod\limits_{i=1}^{p} f_i^{'}\left(u_i\right)\right]} \leq \theta \leq -\frac{1}{\inf\limits_{(u_1,\ldots,u_p)\in D^-} \left[\prod\limits_{i=1}^{p} f_i^{'}\left(u_i\right)\right]},$$

where

$$D^- = \left\{0 \leq u_1, \ldots, u_p \leq 1 : \prod_{i=1}^{p} f_i^{'}\left(u_i\right) < 0\right\}$$

and

$$D^+ = \left\{0 \leq u_1, \ldots, u_p \leq 1 : \prod_{i=1}^{p} f_i^{'}\left(u_i\right) > 0\right\}.$$

The second is given by

$$C\left(u_1, \ldots, u_p\right) = \prod_{i=1}^{p} u_i + \alpha \sum_{1\leq i<j\leq p} f\left(u_i\right) g\left(u_j\right) \prod_{k=1,k\neq i,j}^{p} u_k,$$

where $f, g : [0,1] \to \mathbb{R}$ are non-zero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$ and

$$-\frac{1}{\max\left\{\sum\limits_{1\leq i<j\leq p} f^{'}\left(u_i\right) g^{'}\left(u_j\right)\right\}} \leq \alpha \leq -\frac{1}{\min\left\{\sum\limits_{1\leq i<j\leq p} f^{'}\left(u_i\right) g^{'}\left(u_j\right)\right\}}.$$

The third is given by

$$C\left(u_1, \ldots, u_p\right) = \prod_{i=1}^{p} u_i + \sum_{k=2}^{p} \sum_{1\leq i_1<\cdots<i_k\leq p} \theta_{i_1,\ldots,i_k} \prod_{j=i_1}^{i_k} f_j\left(u_j\right) \prod_{\ell=1,\ell\neq i_1,\ldots,i_k}^{p} u_\ell,$$

where $f_i : [0,1] \to \mathbb{R}$, $i = 1, \ldots, p$ are non-zero absolutely continuous functions such that $f_i(0) = f_i(1) = 0$ and

$$1 + \sum_{k=2}^{p} \sum_{1\leq i_1<\cdots<i_k\leq p} \theta_{i_1,\ldots,i_k} \prod_{j=i_1}^{i_k} f_j\left(u_j\right) \geq 0.$$

2.75 Durante *et al.*'s copula

For a given $(p-1)$-dimensional copula D, Durante *et al.* (2012) have shown that

$$C\left(u_1, \ldots, u_p\right) = u_p D\left(u_1, \ldots, u_{p-1}\right) + f\left(u_p\right) A\left(u_1, \ldots, u_{p-1}\right)$$

is a copula for suitable functions $f : [0,1] \to \mathbb{R}$ and $A : [0,1]^{p-1} \to \mathbb{R}$.

2.76 Fernández Sánchez and Úbeda-Flores's copula

For functions D, E, F and $G : [0,1]^2 \to \mathbb{R}$, Fernández Sánchez and Úbeda-Flores (2012) have shown that

$$C(u_1,u_2) = D(\max(u_1,u_2)) E(\min(u_1,u_2)) + F(\min(u_1,u_2)) G(\max(u_1,u_2))$$

is a copula if and only if C can be expressed as

$$C(u_1,u_2) = A(\max(u_1,u_2)) Z(\min(u_1,u_2)) + \min(u_1,u_2) B(\max(u_1,u_2)), \qquad (12)$$

where $A, B, Z : [0,1] \to \mathbb{R}$ are absolutely continuous functions such that $A(1) = Z(0) = 1$, $B(1) = 1$, $A^{'}(u_2) Z^{'}(u_1) + B^{'}(u_2) \geq 0$ for $u_1 \leq u_2$ in a set of measure $1/2$, and $A^{'}(u_1) Z(u_1) - A(u_1) Z^{'}(u_1) - B(u_1) + u_1 B^{'}(u_1) \leq 0$ in a set of measure 1. Moreover, Fernández Sánchez and Úbeda-Flores (2012) have shown that C in (12) is a copula if and only if $A^{'}(u_1) Z(u_1) - A(u_1) Z^{'}(u_1) - B(u_1) + u_1 B^{'}(u_1) = 0$ for all $0 \leq u_1 \leq 1$.

2.77 Fischer and Köck's copulas

Fischer and Köck (2012) have proposed three further extensions of the FGM copula, see Section 2.3. The first of them is defined by

$$C(u_1,u_2) = u_1 u_2 \left[1 + \theta\left(1 - u_1^{\frac{1}{r}}\right)\left(1 - u_2^{\frac{1}{r}}\right)\right]^r$$

for $r \geq 1$ and $-1 \leq \theta \leq 1$. The second is defined by

$$\begin{aligned} C(u_1,u_2) = \ & 2^{-r}\left\{u_1^{\frac{\alpha}{r}}\left(2u_2^{\frac{1}{r}} - u_2^{\frac{\alpha}{r}}\right)\left[1 + \theta\left(1 - u_1^{\frac{\alpha}{r}}\right)\left(1 - 2u_2^{\frac{1}{r}} + u_2^{\frac{\alpha}{r}}\right)\right]\right. \\ & \left. + u_1^{\frac{\beta}{r}}\left(2u_2^{\frac{1}{r}} - u_2^{\frac{\beta}{r}}\right)\left[1 + \theta\left(1 - u_1^{\frac{\beta}{r}}\right)\left(1 - 2u_2^{\frac{1}{r}} + u_2^{\frac{\beta}{r}}\right)\right]\right\}^r \end{aligned}$$

for $1 \leq \alpha, \beta \leq 2$ and $r \geq 1$. The third and the final one is defined by

$$C(u_1,u_2) = u_1 u_2 \left\{\left[1 + \theta_1\left(1 - u_1^{\frac{1}{r}}\right)\left(1 - u_2^{\frac{1}{r}}\right)\right]^r + \left[1 + \theta_2\left(1 - u_1^{\frac{1}{r}}\right)\left(1 - u_2^{\frac{1}{r}}\right)\right]^r\right\}$$

for $r \geq 1$ and $-1 \leq \theta_1, \theta_2 \leq 1$.

2.78 Li and Fang's copula

Let $\Psi(\cdot)$ denote a cumulative distribution function with its inverse $\Psi^{-1}(t)$ satisfying

$$\int_0^1 \sin\left(\Psi^{-1}(t)\right) dt = 0.$$

Li and Fang (2012) have shown that

$$C(u_1,u_2) = u_1 u_2 + \int_0^{u_1} \sin\left(\Psi^{-1}(t)\right) dt \int_0^{u_2} \sin\left(\Psi^{-1}(t)\right) dt$$

is a valid copula.

2.79 Pougaza and Mohammad-Djafari's copula

Suppose F_i, $i = 1, \ldots, p$ are absolutely continuous one-dimensional cumulative distribution functions. Pougaza and Mohammad-Djafari (2012) have shown that

$$C\left(u_1, \ldots, u_p\right) = \max\left\{\sum_{i=1}^{p} u_i \prod_{j=1, j\neq i}^{p} F_j^{-1}\left(u_j\right) + (1-p)\prod_{i=1}^{p} F_i^{-1}\left(u_i\right), 0\right\}$$

is a valid copula.

2.80 Sanfins and Valle' copula

Let $x = \psi_m(u)$ denote the root of the equation

$$u = x\sum_{j=0}^{m-1}(-1)^j\frac{(\log x)^j}{j!}$$

and let

$$r_{m-1}\left(u_1, \ldots, u_m\right) = \psi_\ell\left(u_\ell\right)\sum_{j=0}^{m-1}(-1)^j\frac{\left[\log\psi_\ell\left(u_\ell\right)\right]^j}{j!}$$

if $\psi_\ell\left(u_\ell\right) = \min\left[\psi_1\left(u_1\right), \ldots, \psi_m\left(u_m\right)\right]$. For every $\left(u_1, \ldots, u_p\right)$ such that $u = \psi_1\left(u_1\right) \geq \cdots \geq \psi_p\left(u_p\right)$, let

$$\mathcal{H}_p\left(u_1, \ldots, u_p\right) = u_p - \psi_p\left(u_p\right)\sum_{j=1}^{p-1}\frac{\left[-\log\psi_j\left(u_j\right)\right]^j}{j!}J_{p-j}\left(-\log\psi_{j+1}\left(u_{j+1}\right), \ldots, -\log\psi_p\left(u_p\right)\right),$$

where J_m is given by the recurrence relation

$$J_m\left(x_1, \ldots, x_m\right) = \sum_{j=0}^{m-1}\frac{x_m^j}{j!} - \sum_{j=0}^{m-1}\frac{x_j^j}{j!}J_{m-j}\left(x_{j+1}, \ldots, x_m\right)$$

for $m \geq 1$ with $J_1 \equiv 1$. Under these assumptions, Sanfins and Valle (2012) have shown that

$$C\left(u_1, \ldots, u_p\right) = \mathcal{H}_p\left(u_1, r_1\left(u_1, u_2\right), r_2\left(u_1, u_2, u_3\right), \ldots, r_{p-1}\left(u_1, \ldots, u_p\right)\right)$$

is a valid copula.

2.81 Bozkurt's copulas

Bozkurt (2013) have proposed four extensions of the FGM copula, see Section 2.3. The first these is defined by

$$C\left(u_1, u_2\right) = \beta u_1 u_2\left[1 + \alpha\left(1-u_1\right)^2\left(1-u_2\right)\right] + (1-\beta)u_1 u_2\left[1 + \alpha\left(1-u_1\right)\left(1-u_2\right)\right]$$

for $0 \leq \beta \leq 1$ and

$$\max\left(-\frac{3\beta}{\beta^2-\beta+1},-1\right) \leq \alpha \leq \min\left(\frac{3\beta}{\beta^2-\beta+1},1\right).$$

The particular case for $\beta = 0$ is the FGM copula. The second is defined by

$$C(u_1,u_2) = \beta u_1 u_2 \left[1+\alpha\left(1-u_1^2\right)(1-u_2)\right] + (1-\beta) u_1 u_2 \left[1+\alpha(1-u_1)(1-u_2)\right]$$

for $0 \leq \beta \leq 1$ and

$$\max\left(-\frac{3\beta}{\beta^2-\beta+1},-\frac{1}{\beta+1}\right) \leq \alpha \leq \min\left(\frac{3\beta}{\beta^2-\beta+1},\frac{1}{\beta+1}\right).$$

The third one is defined by

$$C(u_1,u_2) = \beta u_1 u_2 \left[1+\alpha\left(1-u_1^p\right)\left(1-u_2^p\right)\right] + (1-\beta) u_1 u_2 \left[1+\alpha\left(1-u_1^q\right)\left(1-u_2^q\right)\right]$$

for $p > 0$, $q > 0$, $0 \leq \beta \leq 1$ and

$$\max\left(-1,-\frac{1}{p^2},-\frac{1}{q^2}\right) \leq \alpha \leq \min\left(\frac{1}{p},\frac{1}{q}\right).$$

The fourth and the final one is defined by

$$C(u_1,u_2) = \beta u_1 u_2 \left[1+\alpha(1-u_1)^p(1-u_2)^p\right] + (1-\beta) u_1 u_2 \left[1+\alpha(1-u_1)^q(1-u_2)^q\right]$$

for $p > 1$, $q > 1$, $0 \leq \beta \leq 1$ and

$$-1 \leq \alpha \leq \min\left[\left(\frac{p+1}{p-1}\right)^{p-1},\left(\frac{q+1}{q-1}\right)^{q-1}\right].$$

2.82 Durante *et al.*'s copula

For a given copula D and given $f, g : [0,1] \to \mathbb{R}$, Durante *et al.* (2013) have shown that

$$C(u_1,u_2) = D(u_1,u_2) + f(\max(u_1,u_2))\, g(\min(u_1,u_2))$$

is a valid copula if and only if i) $f(1) = g(0) = 0$; ii) f and g are absolutely continuous functions; iii) $\partial D(u_1,u_2)/\partial u_1 \partial u_2 \geq f^{'}(\max(u_1,u_2))\, g^{'}(\min(u_1,u_2))$ almost everywhere in $[0,1]^2$; iv) if ν denotes the density of the mass distribution of D along the main diagonal of $[0,1]^2$, then $\nu(u) \geq f^{'}(u)g(u) - f(u)g^{'}(u)$ almost every where in $[0,1]$.

2.83 (α,β)-homogeneous copulas

Let α, β be real numbers such that $\max(u_1+u_2-1,0) = W(u_1,u_2) \leq \beta \leq \alpha < 1$. Fernández-Sánchez *et al.* (2013) have defined C to be a (α,β)-homogeneous copula if

$$C(\alpha u_1, \alpha u_2) = \beta C(u_1,u_2)$$

for all $0 \leq u_1, u_2 \leq 1$. Fernández-Sánchez *et al.* (2013) have showed that a given copula C is a (α, β)-homogeneous copula if and only if there exist three copulas, C_A, C_B and C_D, such that

$$C(\alpha u_1, \alpha u_2) = \begin{cases} \beta^i(\alpha-\beta) C_A\left(\frac{u_1}{\alpha^{i+1}}, G_i(u_2)\right) + \left(\frac{\beta}{\alpha}\right)^{i+1} u_1, \\ \qquad \text{if } (u_1, u_2) \in A_i, \\ \\ \beta^i(\alpha-\beta) C_B\left(F_i(u_1), \frac{u_2}{\alpha^{i+1}}\right) + \left(\frac{\beta}{\alpha}\right)^{i+1} u_2, \\ \qquad \text{if } (u_1, u_2) \in B_i, \\ \\ \beta^{i+1} + \beta^i(\alpha-\beta)\left[F_i(u_1) + G_i(u_2)\right] + \beta^i(1-2\alpha+\beta) C_D(K_i(u_1), L_i(u_2)), \\ \qquad \text{if } (u_1, u_2) \in D_i \end{cases}$$

for every $(u_1, u_2) \in A_i \cup B_i \cup D_i$ and for every $i \geq 1$, where

$$\begin{aligned} A_i &= \left\{(u_1, u_2) \in [0,1]^2 : 0 \leq u_1 \alpha^{i+1} \leq u_2 \leq \alpha^i\right\}, \\ B_i &= \left\{(u_1, u_2) \in [0,1]^2 : 0 \leq u_2 \alpha^{i+1} \leq u_1 \leq \alpha^i\right\}, \\ D_i &= \left\{(u_1, u_2) \in [0,1]^2 : \alpha^{i+1} \leq u_1, u_2 \leq \alpha^i\right\}, \\ A_i &= \left\{(u_1, u_2) \in [0,1]^2 : 0 \leq u_1 \alpha^{i+1} \leq u_2 \leq \alpha^i\right\}, \\ F_i(u_1) &= F\left(\frac{u_1 - \alpha^{i+1}}{(1-\alpha)\alpha^i}\right), \ K_i(u_1) = K\left(\frac{u_1 - \alpha^{i+1}}{(1-\alpha)\alpha^i}\right), \quad u_1 \in \left[\alpha^{i+1}, \alpha^i\right], \\ G_i(u_1) &= G\left(\frac{u_2 - \alpha^{i+1}}{(1-\alpha)\alpha^i}\right), \ L_i(u_1) = L\left(\frac{u_2 - \alpha^{i+1}}{(1-\alpha)\alpha^i}\right), \quad u_2 \in \left[\alpha^{i+1}, \alpha^i\right], \end{aligned}$$

where

$$\begin{aligned} F(u_1) &= \frac{C(\alpha + (1-\alpha)u_1, \alpha) - \beta}{\alpha - \beta}, \ K(u_1) = \frac{(1-\alpha)u_1 - (\alpha-\beta)F(u_1)}{1 - 2\alpha + \beta}, \\ G(u_2) &= \frac{C(\alpha, \alpha + (1-\alpha)u_2) - \beta}{\alpha - \beta}, \ L(u_2) = \frac{(1-\alpha)u_2 - (\alpha-\beta)G(u_2)}{1 - 2\alpha + \beta}. \end{aligned}$$

2.84 Kolesárová *et al.*'s copula

For a given copula D, Kolesárová *et al.* (2013) have shown that

$$C(u_1, u_2) = D(u_1, u_2)\left[u_1 + u_2 - D(u_1, u_2)\right]$$

is a copula, where $u_1 + u_2 - D(u_1, u_2)$ is the dual of the copula D.

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