Application: Generalized Financial Ratios

Ratios of random variables are prevalent in finance. Examples include: current ratio, sales margin, changes in capital employed, interest cover, liabilities ratio and financial leverage ratio. Here, we derive the exact distribution of the ratio $X/(X+Y)$ when X and Y are independent generalized Pareto random variables, Pareto distribution being the first and the most popular distribution used in finance.

1 Introduction

Ratios of random variables arise most frequently in finance. There are many financial indices that take the form of ratios. Some of the most commonly known examples are:

1. Current ratio defined by Current assets (X)/Current liabilities (Y).
2. Sales margin defined by (Sales (X) - Costs (Y))/Sales (X).
3. Changes in capital employed defined by (Closing capital (Y) - Opening capital (X))/Opening capital (X).
4. Interest cover defined by (Earnings (X) + Interests paid (Y))/Earnings (X).
5. Liabilities ratio defined by Liabilities (X)/(Equity (Y) + Liabilities (X)).
6. Financial leverage ratio defined by Liabilities (X)/(Total capital (Y) - Liabilities (X)).

Each of the above ratios can be re–expressed as a function of $W = X/(X+Y)$, which takes values between 0 and 1. For example, the current ratio can be re–expressed as $W/(1-W)$ and the sales margin as $2 - 1/W$.

In this note, we consider the probability distribution of $W = X/(X+Y)$. We take X and Y to be independent generalized Pareto random variables specified by the probability density functions (pdfs)

$$f_X(x) = \frac{1}{k}\left(1 - \frac{cx}{k}\right)^{1/c-1} \tag{1}$$

and

$$f_Y(y) = \frac{1}{m}\left(1 - \frac{dy}{m}\right)^{1/d-1}, \tag{2}$$

respectively, for $x > 0$ (if $c \leq 0$), $0 < x < k/c$ (if $c > 0$), $y > 0$ (if $d \leq 0$), $0 < y < m/d$ (if $d > 0$), $k > 0$, $m > 0$, $-\infty < c < \infty$ and $-\infty < d < \infty$. The Pareto distribution is chosen because it is the first and the most popular distribution used in the actuarial sciences and finance. The recent book by Kleiber and Kotz (2003) describes it as the pillar of statistical "size" distributions.

The exact expressions for the distribution of $W = X/(X+Y)$ are given in Section 2. In Section 3, we provide a tabulation of the associated percentage points. The analytical calculations involve the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. The properties of the Gauss hypergeometric function can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Exact distribution of the ratio

We derive various representations for the pdf of $W = X/(X+Y)$. Theorem 1 expresses the pdf explicitly in terms of the Gauss hypergeometric function.

Theorem 1 *Suppose X and Y are independent generalized Pareto random variables with pdfs (1) and (2), respectively. The pdf of $W = X/(X+Y)$ can be expressed as one of the following:*

1. *if $c \leq 0$ and $d \leq 0$ then*

$$f_W(w) = \frac{k}{mc^2w^2} B\left(2, -\frac{1}{c} - \frac{1}{d}\right) {}_2F_1\left(2, 1-\frac{1}{d}; 2-\frac{1}{c}-\frac{1}{d}; 1-\frac{kd(1-w)}{mcw}\right). \quad (3)$$

2. *if $c \leq 0$ and $d > 0$ then*

$$f_W(w) = \frac{1}{dk(1-w)} B\left(2, \frac{1}{d}\right) {}_2F_1\left(2, 1-\frac{1}{d}; 2+\frac{1}{c}; \frac{mcw}{kd(1-w)}\right). \quad (4)$$

3. *if $c > 0$, $d > 0$ and $w < (k/c)/\{(m/d)+(k/c)\}$ then*

$$f_W(w) = \frac{m}{d^2k(1-w)^2} B\left(2, \frac{1}{d}\right) {}_2F_1\left(2, 1-\frac{1}{c}; 2+\frac{1}{d}; \frac{mcw}{kd(1-w)}\right). \quad (5)$$

4. *if $c > 0$, $d > 0$ and $w > (k/c)/\{(m/d)+(k/c)\}$ then*

$$f_W(w) = \frac{k}{mc^2w^2} B\left(2, \frac{1}{c}\right) {}_2F_1\left(2, 1-\frac{1}{d}; 2+\frac{1}{c}; \frac{kd(1-w)}{mcw}\right). \quad (6)$$

Proof: The joint pdf of $(R, W) = (X+Y, X/R)$ is given by

$$f(r,w) = \frac{r}{km}\left(1-\frac{crw}{k}\right)^{1/c-1}\left(1-\frac{dr(1-w)}{m}\right)^{1/d-1}. \quad (7)$$

The pdf W follows by integrating (7) with respect to all possible values of r. If $c \leq 0$ and $d \leq 0$ then note that we can write

$$\begin{aligned} f_W(w) &= \frac{1}{km}\left(-\frac{cw}{k}\right)^{1/c-1}\left(-\frac{d(1-w)}{m}\right)^{1/d-1} \\ &\quad \times \int_0^\infty r\left(r-\frac{k}{cw}\right)^{1/c-1}\left(r-\frac{m}{d(1-w)}\right)^{1/d-1} dr. \end{aligned} \quad (8)$$

If either $c \leq 0$ and $d > 0$ or $c > 0$, $d > 0$ and $w < (k/c)/\{(m/d)+(k/c)\}$ then

$$\begin{aligned} f_W(w) &= \frac{1}{km}\left(-\frac{cw}{k}\right)^{1/c-1}\left(\frac{d(1-w)}{m}\right)^{1/d-1} \\ &\quad \times \int_0^{m/\{d(1-w)\}} r\left(r-\frac{k}{cw}\right)^{1/c-1}\left(\frac{m}{d(1-w)}-r\right)^{1/d-1} dr. \end{aligned} \quad (9)$$

If $c > 0$, $d > 0$ and $w > (k/c)/\{(m/d) + (k/c)\}$ then

$$\begin{aligned} f_W(w) &= \frac{1}{km}\left(\frac{cw}{k}\right)^{1/c-1}\left(-\frac{d(1-w)}{m}\right)^{1/d-1} \\ &\quad\times \int_0^{k/(cw)} r\left(\frac{k}{cw}-r\right)^{1/c-1}\left(r-\frac{m}{d(1-w)}\right)^{1/d-1} dr. \end{aligned} \tag{10}$$

The result in (3) follows by applying equation (2.2.6.24) in Prudnikov *et al.* (1986, volume 1) to calculate the integral in (8). The results in (4)–(5) follow by applying equation (2.2.6.15) in Prudnikov *et al.* (1986, volume 1) to calculate the integral in (9). Finally, the result in (6) follows by applying equation (2.2.6.15) in Prudnikov *et al.* (1986, volume 1) to calculate the integral in (10). ■

Using special properties of the Gauss hypergeometric function, one can reduce (3)–(6) to elementary forms when $1/c$ and $1/d$ take integer values. This is illustrated in the corollaries below.

Corollary 1 *If $-1/c \geq 1$ and $-1/d \geq 1$ are integers then (3) can be reduced to*

$$f_W(w) = \frac{1}{km}\left(-\frac{cw}{k}\right)^{1/c-1}\left(-\frac{d(1-w)}{m}\right)^{1/d-1}\left\{I\left(-\frac{1}{c}, 1-\frac{1}{d}\right)+\frac{k}{cw}I\left(1-\frac{1}{c}, 1-\frac{1}{d}\right)\right\},$$

where $I(m,n)$ satisfies the recurrence relation

$$\begin{aligned} I(m,n) &= \frac{cdw(1-w)}{(m-1)\{mcw-kd(1-w)\}}\left(-\frac{cw}{k}\right)^{n-1}\left(-\frac{d(1-w)}{m}\right)^{m-1} \\ &\quad -\frac{(m+n-2)cdw(1-w)I(m-1,n)}{(m-1)\{mcw-kd(1-w)\}} \end{aligned}$$

for $m > 1$ and $n > 1$ with the initial values

$$I(1,1) = \frac{cdw(1-w)}{mcw-kd(1-w)}\log\left(\frac{mcw}{kd(1-w)}\right),$$

$$I(m,0) = \frac{1}{m-1}\left(-\frac{m}{d(1-w)}\right)^{1-m}$$

and

$$I(0,n) = \frac{1}{n-1}\left(-\frac{k}{cw}\right)^{1-n}.$$

Corollary 2 *If $-1/c \geq 1$ and $1/d \geq 1$ are integers then (4) can be reduced to*

$$\begin{aligned} f_W(w) &= \frac{(-1)^{1/c-1}}{km}\left(\frac{d(1-w)}{m}\right)^{1/d-1}\sum_{n=0}^{1/d-1}\sum_{j=0}^{n+1}\binom{1/d-1}{n}\binom{n+1}{j}\left(-\frac{d(1-w)}{m}\right)^{n} \\ &\quad\times\left(\frac{k}{cw}\right)^{n-j-1/c+2} D(j,n), \end{aligned}$$

where

$$D(j,n) = \begin{cases} \dfrac{1}{j+1/c}\left[\left\{\dfrac{m}{d(1-w)} - \dfrac{k}{cw}\right\}^{j+1/c} - \left\{-\dfrac{k}{cw}\right\}^{j+1/c}\right], & \text{if } j+1/c \neq 0, \\ \log\left\{\dfrac{m}{d(1-w)} - \dfrac{k}{cw}\right\} - \log\left\{-\dfrac{k}{cw}\right\}, & \text{if } j+1/c = 0. \end{cases}$$

Corollary 3 *If $1/c \geq 1$ and $1/d \geq 1$ are integers then (5) can be reduced to*

$$f_W(w) = \frac{1}{km}\sum_{i=0}^{1/c-1}\sum_{j=0}^{1/d-1}\binom{1/c-1}{i}\binom{1/d-1}{j}\left(-\frac{cw}{k}\right)^i\left(-\frac{d(1-w)}{m}\right)^j\frac{\{m/(d(1-w))\}^{i+j+2}}{i+j+2}$$

for $w < (k/c)/\{(m/d)+(k/c)\}$.

Corollary 4 *If $1/c \geq 1$ and $1/d \geq 1$ are integers then (6) can be reduced to*

$$f_W(w) = \frac{1}{km}\sum_{i=0}^{1/c-1}\sum_{j=0}^{1/d-1}\binom{1/c-1}{i}\binom{1/d-1}{j}\left(-\frac{cw}{k}\right)^i\left(-\frac{d(1-w)}{m}\right)^j\frac{\{k/(cw)\}^{i+j+2}}{i+j+2}$$

for $w > (k/c)/\{(m/d)+(k/c)\}$.

3 Percentiles

In this section, we provide tabulations of percentage points associated with the derived distribution of $W = X/(X+Y)$. These values are obtained by numerically solving the equation

$$\int_0^{w_q} f_W(w)dw = q,$$

where $f_W(\cdot)$ is given by one of (3)–(6). Evidently, this involves computation of the Gauss hypergeometric function and routines for this are widely available. We used the function hypergeom $(\cdot)$ in the algebraic manipulation package, MAPLE. Table 1 below provides the numerical values of w_q for $k=1$, $m=1$, $c=-0.1,-0.2,\ldots,-1$ and $d=c,c-0.1,\ldots,-1$.

Table 1. Percentage points w_q of $W = X/(X+Y)$.

c	d	$q=0.01$	$q=0.05$	$q=0.1$	$q=0.9$	$q=0.95$	$q=0.99$
-0.1	-0.1	0.009	0.046	0.093	0.907	0.954	0.991
-0.1	-0.2	0.008	0.041	0.085	0.906	0.954	0.991
-0.1	-0.3	0.007	0.037	0.077	0.905	0.954	0.991
-0.1	-0.4	0.006	0.033	0.069	0.904	0.954	0.991
-0.1	-0.5	0.005	0.029	0.062	0.904	0.953	0.991
-0.1	-0.6	0.004	0.025	0.056	0.903	0.953	0.991
-0.1	-0.7	0.004	0.021	0.049	0.902	0.953	0.991
-0.1	-0.8	0.003	0.018	0.044	0.901	0.953	0.991
-0.1	-0.9	0.002	0.016	0.038	0.900	0.952	0.991
-0.1	-1	0.002	0.013	0.034	0.899	0.952	0.991

-0.2	-0.2	0.008	0.042	0.086	0.914	0.958	0.992
-0.2	-0.3	0.007	0.037	0.078	0.913	0.958	0.992
-0.2	-0.4	0.006	0.033	0.070	0.912	0.958	0.992
-0.2	-0.5	0.005	0.029	0.063	0.912	0.958	0.992
-0.2	-0.6	0.004	0.025	0.056	0.911	0.958	0.992
-0.2	-0.7	0.004	0.022	0.050	0.910	0.957	0.992
-0.2	-0.8	0.003	0.019	0.044	0.909	0.957	0.992
-0.2	-0.9	0.002	0.016	0.039	0.908	0.957	0.992
-0.2	-1	0.002	0.013	0.034	0.907	0.957	0.992
-0.3	-0.3	0.007	0.037	0.079	0.921	0.963	0.993
-0.3	-0.4	0.006	0.033	0.071	0.920	0.962	0.993
-0.3	-0.5	0.005	0.029	0.064	0.919	0.962	0.993
-0.3	-0.6	0.004	0.025	0.057	0.919	0.962	0.993
-0.3	-0.7	0.004	0.022	0.051	0.918	0.962	0.993
-0.3	-0.8	0.003	0.019	0.045	0.917	0.961	0.993
-0.3	-0.9	0.002	0.016	0.040	0.916	0.961	0.993
-0.3	-1	0.002	0.014	0.035	0.915	0.961	0.993
-0.4	-0.4	0.006	0.033	0.072	0.928	0.967	0.994
-0.4	-0.5	0.005	0.029	0.065	0.927	0.966	0.994
-0.4	-0.6	0.004	0.026	0.058	0.926	0.966	0.994
-0.4	-0.7	0.004	0.022	0.052	0.925	0.966	0.994
-0.4	-0.8	0.003	0.019	0.046	0.924	0.966	0.994
-0.4	-0.9	0.002	0.016	0.040	0.924	0.965	0.994
-0.4	-1	0.002	0.014	0.036	0.923	0.965	0.994
-0.5	-0.5	0.005	0.030	0.066	0.934	0.970	0.995
-0.5	-0.6	0.004	0.026	0.059	0.933	0.970	0.995
-0.5	-0.7	0.004	0.022	0.053	0.933	0.970	0.995
-0.5	-0.8	0.003	0.019	0.047	0.932	0.970	0.995
-0.5	-0.9	0.002	0.016	0.041	0.931	0.969	0.995
-0.5	-1	0.002	0.014	0.036	0.930	0.969	0.995
-0.6	-0.6	0.004	0.026	0.060	0.940	0.974	0.996
-0.6	-0.7	0.004	0.023	0.053	0.939	0.974	0.996
-0.6	-0.8	0.003	0.019	0.047	0.939	0.973	0.996
-0.6	-0.9	0.002	0.017	0.042	0.938	0.973	0.996
-0.6	-1	0.002	0.014	0.037	0.937	0.973	0.995
-0.7	-0.7	0.004	0.023	0.054	0.946	0.977	0.996
-0.7	-0.8	0.003	0.020	0.048	0.945	0.977	0.996
-0.7	-0.9	0.002	0.017	0.043	0.944	0.977	0.996
-0.7	-1	0.002	0.014	0.038	0.944	0.976	0.996
-0.8	-0.8	0.003	0.020	0.049	0.951	0.980	0.997
-0.8	-0.9	0.002	0.017	0.043	0.950	0.980	0.997
-0.8	-1	0.002	0.015	0.038	0.950	0.980	0.997
-0.9	-0.9	0.002	0.017	0.044	0.956	0.983	0.998
-0.9	-1	0.002	0.015	0.039	0.955	0.983	0.998
-1	-1	0.002	0.015	0.040	0.960	0.985	0.998

These numbers can be used to determine the probabilities of the observed financial ratios mentioned in Section 1. Similar tabulations could be easily derived for other values of q, k, m, c and d by using the hypergeom $(\cdot)$ function in MAPLE.

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