1 GARCH models

GARCH$(1, 1)$ is a popular time series model for weakly stationary financial data. It can be specified by

$$X_t = \sigma_t Z_t, \tag{1}$$

where $\{X_t\}$ is the observed financial data, $\{\sigma_t\}$ is a *volatility process* specified by

$$\sigma_i^2 = \omega + \alpha_1 X_{i-1}^2 + \beta_1 \sigma_{i-1}^2$$

and $\{Z_t\}$ is an *innovation process.*

We consider eight different distributions for Z_t: the Gaussian distribution due to de Moivre (1738) and Gauss (1809); the skewed Gaussian distribution due to Azzalini (1985); the Student's t distribution due to Gosset (1908); the skewed Student's t distribution due to Fernandez and Steel (1998); the generalized error distribution due to Subbotin (1923); the skewed generalized error distribution due to Theodossiou (1998); the standardized normal inverse Gaussian distribution due to Barndorff-Nielsen (1977); the asymmetric exponential power distribution due to Zhu and Zinde-Walsh (2009); the asymmetric Student's t distribution due to Zhu and Galbraith (2010).

The first six distributions are commonly used models for the innovation process. They are implemented in standard computer packages for GARCH modeling. See, for example, the R (R Development Core Team, 2013) contributed package fGarch due to Wuertz and Chalabi (2013). The last two distributions are relatively new. We are not aware of any computer package that has implemented these distributions as possible innovation models.

For each distribution for Z_t, we give explicit expressions for $E\left(Z_t\right)$, $E\left(Z_t^2\right)$, $E\left(Z_t^3\right)$, $E\left(Z_t^4\right)$, $\mathrm{VaR}_p\left(Z_t\right)$ and $\mathrm{ES}_p\left(Z_t\right)$.

1.1 Gaussian distribution

If Z_t are independent and identical Gaussian random variables with mean μ and unit variance then

$$\begin{aligned}
&E\left(Z_t\right) = \mu, \\
&E\left(Z_t^2\right) = \mu^2 + 1, \\
&E\left(Z_t^3\right) = \mu^3 + 3\mu, \\
&E\left(Z_t^4\right) = \mu^4 + 6\mu^2 + 3, \\
&\mathrm{VaR}_p\left(Z_t\right) = \mu + \Phi^{-1}(p), \\
&\mathrm{ES}_p\left(Z_t\right) = \mu p + \phi\left(\Phi^{-1}(p)\right),
\end{aligned}$$

where $\phi(\cdot)$ is the probability density function of a standard Gaussian random variable and $\Phi(\cdot)$ is the cumulative distribution function of a standard Gaussian random variable. Gaussian distribution is due to de Moivre (1738) and Gauss (1809).

1.2 Skewed Gaussian distribution

If Z_t are independent and identical skewed Gaussian random variables with location parameter μ and skewness parameter λ then

$$
\begin{aligned}
E\left(Z_t\right) &= \mu + \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{1+\lambda^2}}, \\
E\left(Z_t^2\right) &= 1 + \mu^2 + 2\mu \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{1+\lambda^2}}, \\
E\left(Z_t^3\right) &= \mu^3 + 3\mu^2 \frac{\lambda}{\sqrt{1+\lambda^2}} + 3\mu + \sqrt{\frac{2}{\pi}} \frac{\lambda\left(3+2\lambda^2\right)}{\left(1+\lambda^2\right)^{3/2}}, \\
E\left(Z_t^4\right) &= \mu^4 + 6\mu^3 \frac{\lambda}{\sqrt{1+\lambda^2}} + 6\mu^2 + 4\mu \sqrt{\frac{2}{\pi}} \frac{\lambda\left(3+2\lambda^2\right)}{\left(1+\lambda^2\right)^{3/2}} + 3, \\
\mathrm{ES}_p\left(Z_t\right) &= 2 \int_{-\infty}^{\mathrm{VaR}} x \phi(x-\mu) \Phi\left(\lambda(x-\mu)\right) dx,
\end{aligned}
$$

where $\mathrm{VaR}_p\left(Z_t\right)$ is the root of

$$
\Phi(x-\mu) - 2T\left(x-\mu, \lambda\right) = p,
$$

where $T(h,a)$ is Owen's function defined in Owen (1980). The skewed Gaussian distribution is due to Azzalini (1985). We shall abbreviate this distribution by SNORM.

1.3 Student's t distribution

If Z_t are independent and identical Student's t random variables with location parameter μ and degrees of freedom ν then

$$
\begin{aligned}
E\left(Z_t\right) &= \mu, \\
E\left(Z_t^2\right) &= \mu^2 + \frac{\nu}{\nu-2}, \\
E\left(Z_t^3\right) &= \mu^3 + \frac{3\mu\nu}{\nu-2}, \\
E\left(Z_t^4\right) &= \mu^4 + \frac{6\mu^2\nu}{\nu-2} + \frac{3\nu^2}{(\nu-2)(\nu-4)}, \\
\mathrm{VaR}_p\left(Z_t\right) &= \mu + \sqrt{\nu}\,\mathrm{sign}\left(p-\frac{1}{2}\right) \sqrt{\frac{1}{I_a^{-1}\left(\frac{\nu}{2},\frac{1}{2}\right)} - 1}, \\
\mathrm{ES}_p\left(Z_t\right) &= \mu p + \frac{\sqrt{\nu}\Gamma\left((\nu+1)/2\right)}{(1-\nu)\sqrt{\pi}\Gamma(\nu/2)} \left[1 + \frac{(\mathrm{VaR}-\mu)^2}{\nu}\right]^{\frac{1-\nu}{2}},
\end{aligned}
$$

where $a = 2p$ if $p < 1/2$, $a = 2(1-p)$ if $p \geq 1/2$, and $I_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt / B(a,b)$ is the incomplete beta function ratio and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ is the beta function. The Student's t distribution is due to Gosset (1908).

1.4 Skewed Student's t distribution

If Z_t are independent and identical skewed Student's t random variables with location parameter μ, skewness parameter λ and degrees of freedom ν then

$$
\begin{aligned}
E\left(Z_t\right) &= \mu + \frac{\Gamma\left((\nu-1)/2\right)\left[\gamma^2-\gamma^{-2}\right]}{\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]}, \\
E\left(Z_t^2\right) &= \mu^2 + \frac{2\mu\Gamma\left((\nu-1)/2\right)\left[\gamma^2-\gamma^{-2}\right]}{\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]} + \frac{\sqrt{\pi\nu}\Gamma\left((\nu-2)/2\right)\left[\gamma^3+\gamma^{-3}\right]}{2\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]}, \\
E\left(Z_t^3\right) &= \mu^3 + \frac{3\mu^2\Gamma\left((\nu-1)/2\right)\left[\gamma^2-\gamma^{-2}\right]}{\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]} + \frac{3\mu\sqrt{\pi\nu}\Gamma\left((\nu-2)/2\right)\left[\gamma^3+\gamma^{-3}\right]}{2\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]} \\
&\quad + \frac{\nu\Gamma\left((\nu-3)/2\right)\left[\gamma^4-\gamma^{-4}\right]}{\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]}, \\
E\left(Z_t^4\right) &= \mu^4 + \frac{4\mu^3\Gamma\left((\nu-1)/2\right)\left[\gamma^2-\gamma^{-2}\right]}{\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]} + \frac{6\mu^2\sqrt{\pi\nu}\Gamma\left((\nu-2)/2\right)\left[\gamma^3+\gamma^{-3}\right]}{2\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]} \\
&\quad + \frac{6\mu\nu\Gamma\left((\nu-3)/2\right)\left[\gamma^4-\gamma^{-4}\right]}{\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]} + \frac{3\nu^{3/2}\sqrt{\pi}\Gamma\left((\nu-4)/2\right)\left[\gamma^5+\gamma^{-5}\right]}{4\Gamma\left(\nu/2\right)\left[\gamma+\gamma^{-1}\right]}, \\
\mathrm{VaR}_p\left(Z_t\right) &= \begin{cases} \mu + \sqrt{\gamma^{-2}\nu\left[\left\{I_{2\gamma p}^{-1}\left(\frac{\nu}{2},\frac{1}{2}\right)\right\}^{-1} - 1\right]}, & \text{if } p \leq 1/(2\gamma), \\ \mu + \sqrt{\gamma^{2}\nu\left[\left\{I_{1+\gamma^{-2}-2\gamma^{-1}p}^{-1}\left(\frac{\nu}{2},\frac{1}{2}\right)\right\}^{-1} - 1\right]}, & \text{if } p > 1/(2\gamma), \end{cases} \\
\mathrm{ES}_p\left(Z_t\right) &= \begin{cases} \mu p + \dfrac{\sqrt{\nu}\Gamma\left((\nu+1)/2\right)}{\gamma^2(1-\nu)\sqrt{\pi}\Gamma\left(\nu/2\right)}\left(1+\dfrac{\gamma^2\mathrm{VaR}^2}{\nu}\right)^{\frac{1-\nu}{2}}, & \text{if } \mathrm{VaR} \leq \mu, \\ \mu p + \dfrac{\sqrt{\nu}\Gamma\left((\nu+1)/2\right)}{(1-\nu)\sqrt{\pi}\Gamma\left(\nu/2\right)}\left[\gamma^2\left(1+\dfrac{\mathrm{VaR}^2}{\gamma^2\nu}\right)^{\frac{1-\nu}{2}} - \gamma^2 + \gamma^{-2}\right], & \text{if } \mathrm{VaR} > \mu, \end{cases}
\end{aligned}
$$

where $I_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt/B(a,b)$ is the incomplete beta function ratio and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the beta function. The skewed Student's t distribution is due to Fernandez and Steel (1998). We shall abbreviate this distribution by SSTD0.

1.5 Generalized error distribution

If Z_t are independent and identical generalized error random variables with location parameter μ and shape parameter a then

$$
\begin{aligned}
E\left(Z_t\right) &= \mu, \\
E\left(Z_t^2\right) &= \mu^2 + \frac{a^{2/a-1}\Gamma\left(3/a\right)}{\Gamma\left(1+1/a\right)}, \\
E\left(Z_t^3\right) &= \mu^3 + \frac{3\mu a^{2/a-1}\Gamma\left(3/a\right)}{\Gamma\left(1+1/a\right)},
\end{aligned}
$$

$$E\left(Z_t^4\right) = \mu^4 + \frac{6\mu^2 a^{2/a-1}\Gamma\left(3/a\right)}{\Gamma\left(1+1/a\right)} + \frac{a^{4/a-1}\Gamma\left(5/a\right)}{\Gamma\left(1+1/a\right)},$$

$$\mathrm{VaR}_p\left(Z_t\right) = \begin{cases} \mu - a^{1/a}\left[Q^{-1}\left(\dfrac{1}{a}, 2p\right)\right]^{1/a}, & \text{if } p \leq 1/2, \\ \mu + a^{1/a}\left[Q^{-1}\left(\dfrac{1}{a}, 2(1-p)\right)\right]^{1/a}, & \text{if } p > 1/2, \end{cases}$$

$$\mathrm{ES}_p\left(Z_t\right) = \begin{cases} \mu p - \dfrac{a^{1/a}}{2\Gamma(1/a)}\Gamma\left(\dfrac{1}{a}, \dfrac{(\mu-\mathrm{VaR})^a}{a}\right), & \text{if VaR} \leq \mu, \\ \mu p - \dfrac{a^{1/a}}{2} + \dfrac{a^{1/a}}{2\Gamma(1/a)}\gamma\left(\dfrac{1}{a}, \dfrac{(\mathrm{VaR}-\mu)^a}{a}\right), & \text{if VaR} > \mu, \end{cases}$$

where $Q(a,x) = \int_x^\infty t^{a-1}\exp\left(-t\right)dt/\Gamma(a)$ is the regularized complementary incomplete gamma function, $\gamma(a,x) = \int_0^x t^{a-1}\exp\left(-t\right)dt$ is the incomplete gamma function, and $\Gamma(a,x) = \int_x^\infty t^{a-1}\exp\left(-t\right)dt$ is the complementary incomplete gamma function. The generalized error distribution is due to Subbotin (1923). We shall abbreviate this distribution by GED.

1.6 Skewed generalized error distribution

If Z_t are independent and identical generalized error random variables with location parameter μ and shape parameter a then

$$\begin{aligned}
E\left(Z_t\right) &= \mu - \delta + \frac{C\theta^2}{k}\left[-(1-\lambda)^2 + (1+\lambda)^2\right]\Gamma\left(\frac{2}{k}\right), \\
E\left(Z_t^2\right) &= (\mu-\delta)^2 + \frac{2C(\mu-\delta)\theta^2}{k}\left[-(1-\lambda)^2 + (1+\lambda)^2\right]\Gamma\left(\frac{2}{k}\right) \\
&\quad + \frac{C\theta^3}{k}\left[(1-\lambda)^3 + (1+\lambda)^3\right]\Gamma\left(\frac{3}{k}\right), \\
E\left(Z_t^3\right) &= (\mu-\delta)^3 + \frac{3C(\mu-\delta)^2\theta^2}{k}\left[-(1-\lambda)^2 + (1+\lambda)^2\right]\Gamma\left(\frac{2}{k}\right) \\
&\quad + \frac{3C(\mu-\delta)\theta^3}{k}\left[(1-\lambda)^3 + (1+\lambda)^3\right]\Gamma\left(\frac{3}{k}\right) \\
&\quad + \frac{C\theta^4}{k}\left[-(1-\lambda)^4 + (1+\lambda)^4\right]\Gamma\left(\frac{4}{k}\right), \\
E\left(Z_t^4\right) &= (\mu-\delta)^4 + \frac{4C(\mu-\delta)^3\theta^2}{k}\left[-(1-\lambda)^2 + (1+\lambda)^2\right]\Gamma\left(\frac{2}{k}\right) \\
&\quad + \frac{6C(\mu-\delta)^2\theta^3}{k}\left[(1-\lambda)^3 + (1+\lambda)^3\right]\Gamma\left(\frac{3}{k}\right) \\
&\quad + \frac{C(\mu-\delta)\theta^4}{k}\left[-(1-\lambda)^4 + (1+\lambda)^4\right]\Gamma\left(\frac{4}{k}\right) \\
&\quad + \frac{C\theta^5}{k}\left[(1-\lambda)^5 + (1+\lambda)^5\right]\Gamma\left(\frac{5}{k}\right),
\end{aligned}$$

$$\mathrm{VaR}_p\left(Z_t\right)=\begin{cases}\mu-\delta-(1+\lambda)\theta\left[Q^{-1}\left(\dfrac{1}{k},\dfrac{2p}{1+\lambda}\right)\right]^{1/k}, & \text{if } p\le \frac{1+\lambda}{2},\\ \\ \mu-\delta+(1-\lambda)\theta\left[Q^{-1}\left(\dfrac{1}{k},\dfrac{2(1-p)}{1-\lambda}\right)\right]^{1/k}, & \text{if } p> \frac{1+\lambda}{2},\end{cases}$$

$$\mathrm{ES}_p\left(Z_t\right)=\begin{cases}-\dfrac{C(1+\lambda)^2\theta^2}{k}\Gamma\left(\dfrac{2}{k},\dfrac{(\mu-\mathrm{VaR}-\delta)^2}{(1+\lambda)^k\theta^k}\right), \\ \qquad\qquad \text{if VaR}\le\mu-\delta,\\ \\ -\dfrac{C(1+\lambda)^2\theta^2}{k}\Gamma\left(\dfrac{2}{k}\right)+\dfrac{C(1-\lambda)^2\theta^2}{k}\gamma\left(\dfrac{2}{k},\dfrac{(\mathrm{VaR}-\mu+\delta)^2}{(1-\lambda)^k\theta^k}\right), \\ \qquad\qquad \text{if VaR}>\mu-\delta,\end{cases}$$

where $C=k/\left\{2\theta\Gamma(1/k)\right\}$, $\theta=\sqrt{\Gamma(1/k)/\Gamma(3/k)}/S(\lambda)$, $\delta=2\lambda A/S(\lambda)$, $S(\lambda)=\sqrt{1+3\lambda^2-4A^2\lambda^2}$, and $A=\Gamma(2/k)/\sqrt{\Gamma(1/k)\Gamma(3/k)}$. The skewed generalized error distribution is due to Theodossiou (1998). We shall abbreviate this distribution by SGED.

1.7 Standardized normal inverse Gaussian distribution

If Z_t are independent and identical standardized normal inverse Gaussian random variables then

$$\begin{aligned}
E\left(Z_t\right)&=\mu+\frac{\beta}{\gamma},\\
E\left(Z_t^2\right)&=\left(\mu+\frac{\beta}{\gamma}\right)^2+\frac{\alpha^2}{\gamma^2},\\
E\left(Z_t^3\right)&=\left(\mu+\frac{\beta}{\gamma}\right)^3+3\left(\mu+\frac{\beta}{\gamma}\right)\frac{\alpha^2}{\gamma^2}+\frac{3\alpha^2\beta}{\gamma^{7/2}},\\
E\left(Z_t^4\right)&=\frac{3\alpha^4}{\gamma^5}\left(\gamma+1+\frac{4\beta^2}{\alpha^2}\right)+4\mu\left(\mu^2+\frac{3\alpha^2}{\gamma^2}\right)\left(\mu+\frac{\beta}{\gamma}\right)+\frac{12\alpha^2\beta\mu}{\gamma^{7/2}}+4\mu\left(\mu+\frac{\beta}{\gamma}\right)^3\\
&\quad-\frac{6\alpha^2\mu^2}{\gamma^2}-6\mu^2\left(\mu+\frac{\beta}{\gamma}\right)^2,\\
\mathrm{ES}_p\left(Z_t\right)&=\frac{\alpha}{\pi}\int_{-\infty}^{\mathrm{VaR}}\frac{K_1\left(\alpha\sqrt{1+(x-\mu)^2}\right)}{\sqrt{1+(x-\mu)^2}}\exp\left(\beta x+\gamma\right)dx,
\end{aligned}$$

where $\gamma=\sqrt{\alpha^2-\beta^2}$, $K_1(\cdot)$ is the modified Bessel function of the second kind of order one and $\mathrm{VaR}_p\left(Z_t\right)$ is the root of

$$\int_{-\infty}^{x}\frac{K_1\left(\alpha\sqrt{1+(y-\mu)^2}\right)}{\sqrt{1+(y-\mu)^2}}\exp\left(\beta y+\gamma\right)dy=p.$$

The normal inverse Gaussian distribution is due to Barndorff-Nielsen (1977). We shall abbreviate this distribution by SNIG.

1.8 Asymmetric exponential power distribution

If Z_t are independent and identical asymmetric exponential power random variables then

$$
\begin{aligned}
E\left(Z_t\right) &= \mu + \frac{(1-\alpha)^2 H_1\left(p_2\right)}{B} - \frac{\alpha^2 H_1\left(p_1\right)}{B}, \\
E\left(Z_t^2\right) &= \mu^2 + \frac{2\mu(1-\alpha)^2 H_1\left(p_2\right)}{B} - \frac{2\mu\alpha^2 H_1\left(p_1\right)}{B} + \frac{(1-\alpha)^3 H_2\left(p_2\right)}{B^2} + \frac{\alpha^3 H_2\left(p_1\right)}{B^2}, \\
E\left(Z_t^3\right) &= \mu^3 + \frac{3\mu^2(1-\alpha)^2 H_1\left(p_2\right)}{B} - \frac{3\mu^2\alpha^2 H_1\left(p_1\right)}{B} + \frac{3\mu(1-\alpha)^3 H_2\left(p_2\right)}{B^2} \\
&\quad + \frac{3\mu\alpha^3 H_2\left(p_1\right)}{B^2} + \frac{(1-\alpha)^4 H_3\left(p_2\right)}{B^3} - \frac{\alpha^4 H_3\left(p_1\right)}{B^3}, \\
E\left(Z_t^4\right) &= \mu^4 + \frac{4\mu^3(1-\alpha)^2 H_1\left(p_2\right)}{B} - \frac{4\mu^3\alpha^2 H_1\left(p_1\right)}{B} + \frac{6\mu^2(1-\alpha)^3 H_2\left(p_2\right)}{B^2} \\
&\quad + \frac{6\mu^2\alpha^3 H_2\left(p_1\right)}{B^2} + \frac{4\mu(1-\alpha)^4 H_3\left(p_2\right)}{B^3} - \frac{4\mu\alpha^4 H_3\left(p_1\right)}{B^3} \\
&\quad + \frac{(1-\alpha)^5 H_4\left(p_2\right)}{B^4} + \frac{\alpha^5 H_4\left(p_1\right)}{B^4}, \\
\mathrm{VaR}_p\left(Z_t\right) &= \begin{cases} \mu - 2\alpha^* \left[p_1 R^{-1}\left(\dfrac{1}{p_1}, 1 - \dfrac{p}{\alpha}\right)\right]^{1/p_1}, & \text{if } p \le \alpha, \\[2ex] \mu - 2\left(1-\alpha^*\right) \left[p_2 R^{-1}\left(\dfrac{1}{p_2}, 1 - \dfrac{1-p}{1-\alpha}\right)\right]^{1/p_2}, & \text{if } p > \alpha, \end{cases} \\
\mathrm{ES}_p\left(Z_t\right) &= \begin{cases} \mu p - 2\alpha^* C\left(p_1\right) \dfrac{1 - R\left(\frac{2}{p_1}, \frac{1}{p_1}\left|\frac{\mathrm{VaR}-\mu}{2\alpha^*}\right|^{p_1}\right)}{1 - R\left(\frac{1}{p_1}, \frac{1}{p_1}\left|\frac{\mathrm{VaR}-\mu}{2\alpha^*}\right|^{p_1}\right)}, \\ \qquad \text{if VaR} \le \mu, \\[2ex] \mu p - \dfrac{2\alpha\alpha^* C\left(p_1\right) - 2(1-\alpha)\left(1-\alpha^*\right) C\left(p_2\right) R\left(\frac{2}{p_2}, \frac{1}{p_2}\left|\frac{\mathrm{VaR}-\mu}{2(1-\alpha^*)}\right|^{p_2}\right)}{\alpha + (1-\alpha) R\left(\frac{1}{p_2}, \frac{1}{p_2}\left|\frac{\mathrm{VaR}-\mu}{2(1-\alpha^*)}\right|^{p_2}\right)}, \\ \qquad \text{if VaR} > \mu, \end{cases}
\end{aligned}
$$

where $R(a,x) = \int_0^x t^{a-1}\exp(-t)\,dt/\Gamma(a)$ is the regularized incomplete gamma function, $K(p) = 1/\left\{2p^{1/p}\Gamma(1+1/p)\right\}$, $\alpha^* = \alpha K\left(p_1\right)/\left\{\alpha K\left(p_1\right) + (1-\alpha)K\left(p_2\right)\right\}$, $B = \alpha K\left(p_1\right) + (1-\alpha)K\left(p_2\right)$, $H_r(p) = p^r\Gamma\left((r+1)/p\right)/\Gamma^{r+1}(1/p)$ and $C(p) = p^{1/p}\Gamma(2/p)/\Gamma(1/p)$. The asymmetric exponential power distribution is due to Zhu and Zinde-Walsh (2009). We shall abbreviate this distribution by AEPD.

1.9 Skewed exponential power distribution

If Z_t are independent and identical skewed exponential power random variables then

$$
\begin{aligned}
E\left(Z_t\right) &= \mu + \frac{(1-2\alpha) H_1(p)}{B}, \\
E\left(Z_t^2\right) &= \mu^2 + \frac{2\mu(1-2\alpha) H_1(p)}{B} + \frac{\left[(1-\alpha)^3 + \alpha^3\right] H_2(p)}{B^2},
\end{aligned}
$$

$$E\left(Z_t^3\right) = \mu^3 + \frac{3\mu^2(1-2\alpha)H_1(p)}{B} + \frac{3\mu\left[(1-\alpha)^3+\alpha^3\right]H_2(p)}{B^2} + \frac{\left(1-4\alpha+6\alpha^2-4\alpha^3\right)H_3(p)}{B^3},$$

$$E\left(Z_t^4\right) = \mu^4 + \frac{4\mu^3(1-2\alpha)H_1(p)}{B} + \frac{6\mu^2\left[(1-\alpha)^3+\alpha^3\right]H_2(p)}{B^2} + \frac{4\mu\left(1-4\alpha+6\alpha^2-4\alpha^3\right)H_3(p)}{B^3} + \frac{\left[(1-\alpha)^5+\alpha^5\right]H_4(p)}{B^4},$$

$$\mathrm{VaR}_p\left(Z_t\right) = \begin{cases} \mu - 2\alpha\left[pR^{-1}\left(\dfrac{1}{p}, 1-\dfrac{p}{\alpha}\right)\right]^{1/p}, & \text{if } p \leq \alpha, \\ \mu - 2(1-\alpha)\left[pR^{-1}\left(\dfrac{1}{p}, 1-\dfrac{1-p}{1-\alpha}\right)\right]^{1/p}, & \text{if } p > \alpha, \end{cases}$$

$$\mathrm{ES}_p\left(Z_t\right) = \begin{cases} \mu p - 2\alpha C(p)\dfrac{1-R\left(\frac{2}{p}, \frac{1}{p}\left|\frac{\mathrm{VaR}-\mu}{2\alpha}\right|^p\right)}{1-R\left(\frac{1}{p}, \frac{1}{p}\left|\frac{\mathrm{VaR}-\mu}{2\alpha}\right|^p\right)}, \\ \quad \text{if VaR} \leq \mu, \\ \mu p - \dfrac{2\alpha^2 C(p) - 2(1-\alpha)^2 C(p) R\left(\frac{2}{p}, \frac{1}{p}\left|\frac{\mathrm{VaR}-\mu}{2(1-\alpha)}\right|^p\right)}{\alpha + (1-\alpha)R\left(\frac{1}{p}, \frac{1}{p}\left|\frac{\mathrm{VaR}-\mu}{2(1-\alpha)}\right|^p\right)}, \\ \quad \text{if VaR} > \mu. \end{cases}$$

The skewed exponential power distribution is due to Zhu and Zinde-Walsh (2009). We shall abbreviate this distribution by SEPD.

1.10 Asymmetric Student's t distribution

If Z_t are independent and identical asymmetric Student's t random variables then

$$E\left(Z_t\right) = \mu - 2\alpha\alpha^* H_1\left(\nu_1\right) + 2(1-\alpha)\left(1-\alpha^*\right) H_1\left(\nu_2\right),$$

$$E\left(Z_t^2\right) = \mu^2 - 4\mu\alpha\alpha^* H_1\left(\nu_1\right) + 4\mu(1-\alpha)\left(1-\alpha^*\right) H_1\left(\nu_2\right) + 4\alpha\left(\alpha^*\right)^2 H_2\left(\nu_1\right) + 4(1-\alpha)\left(1-\alpha^*\right)^2 H_2\left(\nu_2\right),$$

$$E\left(Z_t^3\right) = \mu^3 - 6\mu^2\alpha\alpha^* H_1\left(\nu_1\right) + 6\mu^2(1-\alpha)\left(1-\alpha^*\right) H_1\left(\nu_2\right) + 12\mu\alpha\left(\alpha^*\right)^2 H_2\left(\nu_1\right) + 12\mu(1-\alpha)\left(1-\alpha^*\right)^2 H_2\left(\nu_2\right) - 8\alpha\left(\alpha^*\right)^3 H_3\left(\nu_1\right) + 8(1-\alpha)\left(1-\alpha^*\right)^3 H_3\left(\nu_2\right),$$

$$E\left(Z_t^4\right) = \mu^4 - 8\mu^3\alpha\alpha^* H_1\left(\nu_1\right) + 8\mu^3(1-\alpha)\left(1-\alpha^*\right) H_1\left(\nu_2\right) + 24\mu^2\alpha\left(\alpha^*\right)^2 H_2\left(\nu_1\right) + 24\mu^2(1-\alpha)\left(1-\alpha^*\right)^2 H_2\left(\nu_2\right) - 24\mu\alpha\left(\alpha^*\right)^3 H_3\left(\nu_1\right) + 24\mu(1-\alpha)\left(1-\alpha^*\right)^3 H_3\left(\nu_2\right) + 16\alpha\left(\alpha^*\right)^4 H_4\left(\nu_1\right) + 16(1-\alpha)\left(1-\alpha^*\right)^4 H_4\left(\nu_2\right),$$

$$\mathrm{VaR}_p\left(Z_t\right) = \mu + 2\alpha^* S_{\nu_1}^{-1}\left(\frac{\min(p,\alpha)}{2\alpha}\right) + 2\left(1-\alpha^*\right) S_{\nu_2}^{-1}\left(\frac{\max(p,\alpha)+1-2\alpha}{2(1-\alpha)}\right),$$

$$\mathrm{ES}_p\left(Z_t\right) = \mu p - \frac{4B}{p}\left\{\frac{\left(\alpha^*\right)^2\nu_1}{\nu_1-1}\left\{1+\frac{1}{\nu_1}\left[\frac{\min\left(\mathrm{VaR}-\mu, 0\right)}{2\alpha^*}\right]^2\right\}^{\frac{1-\nu_1}{2}} - \frac{\left(1-\alpha^*\right)^2\nu_2}{\nu_2-1}\right.$$

$$+\frac{(1-\alpha^*)^2\nu_2}{\nu_2-1}\left\{1+\frac{1}{\nu_2}\left[\frac{\max(\mathrm{VaR}-\mu,0)}{2(1-\alpha^*)}\right]^2\right\}^{\frac{1-\nu_2}{2}}\Bigg\},$$

where $S_\nu(\cdot)$ is the cumulative distribution function of a Student's t random variable with ν degrees of freedom, $K(\nu)=\Gamma((\nu+1)/2)/\{\sqrt{\pi\nu}\Gamma(\nu/2)\}$, $\alpha^*=\alpha K(\nu_1)/\{\alpha K(\nu_1)+(1-\alpha)K(\nu_2)\}$, $B=\alpha K(\nu_1)+(1-\alpha)K(\nu_2)$, and $H_r(\nu)=\sqrt{\nu^r/\pi}\Gamma((r+1)/2)\Gamma((\nu-r)/2)/\Gamma(\nu/2)$. The asymmetric Student's t distribution is due to Zhu and Galbraith (2010). We shall abbreviate this distribution by AST.

1.11 Skewed Student's t distribution

If Z_t are independent and identical skewed Student's t random variables then

$$\begin{aligned}
E(Z_t) &= \mu+2(1-2\alpha)H_1(\nu),\\
E(Z_t^2) &= \mu^2+4\mu(1-2\alpha)H_1(\nu)+4\left[\alpha^3+(1-\alpha)^3\right]H_2(\nu),\\
E(Z_t^3) &= \mu^3+6\mu^2(1-2\alpha)H_1(\nu)+12\mu\left[\alpha^3+(1-\alpha)^3\right]H_2(\nu)\\
&\quad +8\left(1-4\alpha+6\alpha^2-4\alpha^3\right)H_3(\nu),\\
E(Z_t^4) &= \mu^4+8\mu^3(1-2\alpha)H_1(\nu)+24\mu^2\left[\alpha^3+(1-\alpha)^3\right]H_2(\nu)\\
&\quad +24\mu\left(1-4\alpha+6\alpha^2-4\alpha^3\right)H_3(\nu)\\
&\quad +16\left[\alpha^5+(1-\alpha)^5\right]H_4(\nu),\\
\mathrm{VaR}_p(Z_t) &= \mu+2\alpha S_\nu^{-1}\left(\frac{\min(p,\alpha)}{2\alpha}\right)+2(1-\alpha)S_\nu^{-1}\left(\frac{\max(p,\alpha)+1-2\alpha}{2(1-\alpha)}\right),\\
\mathrm{ES}_p(Z_t) &= \mu p-\frac{4B}{p}\Bigg\{\frac{\alpha^2\nu}{\nu-1}\left\{1+\frac{1}{\nu}\left[\frac{\min(\mathrm{VaR}-\mu,0)}{2\alpha}\right]^2\right\}^{\frac{1-\nu}{2}}-\frac{(1-\alpha)^2\nu}{\nu-1}\\
&\quad +\frac{(1-\alpha)^2\nu}{\nu-1}\left\{1+\frac{1}{\nu}\left[\frac{\max(\mathrm{VaR}-\mu,0)}{2(1-\alpha)}\right]^2\right\}^{\frac{1-\nu}{2}}\Bigg\}.
\end{aligned}$$

The skewed Student's t distribution is due to Zhu and Galbraith (2010). We shall abbreviate this distribution by SSTD.

2 Data application

2.1 The data

The data we consider are daily stock market prices of five popular commodities: Cocoa bean, Brent crude oil, West Texas intermediate crude oil, Gold and Silver. The data cover the period from 12th of March 1993 to 13th of March 2013. The data were obtained from the database Datastream.

According to Wikipedia, Cocoa bean is "the dried and fully fermented fatty bean of Theobroma cacao, from which cocoa solids and cocoa butter are extracted. They are the basis of chocolate, as well as many Mesoamerican foods such as mole sauce and tejate".

According to Wikipedia, Brent crude oil is a "major trading classification of sweet light crude oil comprising Brent Blend, Forties Blend, Oseberg and Ekofisk crudes (also known as the BFOE Quotation). Brent Crude is sourced from the North Sea. The Brent Crude oil marker is also known as Brent Blend, London Brent and Brent petroleum".

According to Wikipedia, West Texas intermediate crude oil is a " grade of crude oil used as a benchmark in oil pricing. This grade is described as light because of its relatively low density, and sweet because of its low sulfur content. It is the underlying commodity of Chicago Mercantile Exchange's oil futures contracts. The price of WTI is often referenced in news reports on oil prices, alongside the price of Brent crude from the North Sea. Other important oil markers include the Dubai Crude, Oman Crude, and the OPEC Reference Basket. WTI is lighter and sweeter than Brent, and considerably lighter and sweeter than Dubai or Oman".

According to Wikipedia, Gold is a "chemical element with the symbol Au and atomic number 79. It is a dense, soft, malleable, and ductile metal with a bright yellow color and luster that is considered attractive, which is maintained without tarnishing in air or water. Chemically, gold is a transition metal and a group 11 element. It is one of the least reactive chemical elements, solid under standard conditions. The metal therefore occurs often in free elemental (native) form, as nuggets or grains in rocks, in veins and in alluvial deposits. Less commonly, it occurs in minerals as gold compounds, usually with tellurium".

According to Wikipedia, Silver is a "chemical element with the chemical symbol Ag and atomic number 47. A soft, white, lustrous transition metal, it possesses the highest electrical conductivity of any element and the highest thermal conductivity of any metal. The metal occurs naturally in its pure, free form (native silver), as an alloy with gold and other metals, and in minerals such as argentite and chlorargyrite. Most silver is produced as a byproduct of copper, gold, lead, and zinc refining".

Following common practice, the data were transformed by taking logarithms and then first order differences. The histograms of the five transformed data sets are shown in Figure 1. Each histogram appears more or less symmetric about zero.

[Figure 1 about here.]

Some basic statistics of the transformed data sets are summarized in Table 1. The basic statistics summarized are minimum, first quartile, median, mean, third quartile, maximum, standard deviation, coefficient of variation, skewness, kurtosis, inter quartile range and range.

[Table 1 about here.]

The minimum value for each data set is negative. It is smallest for Cocoa bean and largest for Gold. The first quartile value for each data set is also negative. It is smallest West Texas intermediate crude oil and largest for Gold. The median is exactly zero for Cocoa bean, Gold and Silver. The median is largest for West Texas intermediate crude oil. The mean is smallest for Cocoa bean and largest for Silver. The third quartile is smallest for Gold and largest for West Texas intermediate crude oil. The maximum is smallest for Gold and largest for West Texas intermediate crude oil. The standard deviation is smallest for Gold and largest for West Texas intermediate crude oil. The coefficient of variation is smallest for Gold and largest for Cocoa bean.

The Cocoa bean data are positively skewed. The remaining data (Brent crude oil, West

Texas intermediate crude oil, Gold and Silver) are negatively skewed. The smallest of the negative skewness is for West Texas intermediate crude oil. The largest is for Silver.

Each kurtosis value is significantly greater than three, the kurtosis value corresponding to the normal distribution. The smallest kurtosis is for Brent crude oil. The largest is for Cocoa bean.

The inter quartile range is smallest for Gold and largest for West Texas intermediate crude oil. The range is smallest for Gold and largest for Cocoa bean.

[Table 2 about here.]

We now test for normality of each data set. Table 2 gives the p-values from the Anderson-Darling test (Anderson and Darling, 1954), the Cramer-von Mises test, the Kolmogorov-Smirnov test, the Pearson chi-square test, the Jarque-Bera test (Jarque and Bera, 1980), the Geary test (Geary, 1947) and the data driven smooth test. We can see that none of the data sets follow the normal distribution. If the p-values are taken to measure of the degree of non-normality then we can see that the degree of non-normality is largest for Cocoa bean, second largest for Silver, third largest for Gold, fourth largest for West Texas intermediate crude oil and the smallest for Brent crude oil.

2.2 Results

All of the distributions in Section 1 were fitted to each of the data sets discussed. The method of maximum likelihood was used for parameter estimation. The function optimize in R (R Development Core Team, 2013) was used for maximizing the likelihood function.

[Tables 3, 4, 5, 6 and 7 about here.]

Table 3 gives parameter estimates, log-likelihood values and Akaike information criterion (AIC) values for models fitted to Cocoa bean data. Table 4 gives parameter estimates, log-likelihood values and AIC values for models fitted to Brent crude oil data. Table 5 gives parameter estimates, log-likelihood values and AIC values for models fitted to West Texas intermediate crude oil data. Table 6 gives parameter estimates, log-likelihood values and AIC values for models fitted to Gold data. Table 7 gives parameter estimates, log-likelihood values and AIC values for models fitted to Silver data.

According to the AIC values in Table 3, the best fitting model for Cocoa bean data is the asymmetric Student's t distribution. By comparing the likelihood values of the AST distribution ($\log L = 15479.6$) and the SSTD ($\log L = 15148.2$) by the likelihood ratio test, we see that the degree of freedom parameters, ν_1 and ν_2, are significantly different. The right tail of the returns is heavier. The left tail of the returns is lighter.

According to the AIC values in Table 4, the best fitting model for Brent crude oil data is the asymmetric exponential power distribution. By comparing the likelihood values of the AEPD ($\log L = 13110.5$) and the SEPD ($\log L = 13095.5$) by the likelihood ratio test, we see that the shape parameters, p_1 and p_2, are significantly different. The left tail of the returns is heavier. The right tail of the returns is lighter.

According to the AIC values in Table 5, the best fitting model for West Texas intermediate crude oil data is the asymmetric exponential power distribution. By comparing the likelihood values of the AEPD ($\log L = 12755.1$) and the SEPD ($\log L = 12736.8$) by the likelihood ratio test, we see that the shape parameters, p_1 and p_2, are significantly different. The left tail of the returns is heavier. The right tail of the returns is lighter.

According to the AIC values in Table 6, the best fitting model for Gold data is the asymmetric exponential power distribution. By comparing the likelihood values of the AEPD ($\log L = 17787.7$) and the SEPD ($\log L = 17785.2$) by the likelihood ratio test, we see that the shape parameters, p_1 and p_2, are significantly different. The left tail of the returns is heavier. The right tail of the returns is lighter.

According to the AIC values in Table 7, the best fitting model for Silver data is the skewed exponential power distribution, the particular of the asymmetric exponential power distribution for $p_1 = p_2$. By comparing the likelihood values of the AEPD ($\log L = 14027.8$) and the SEPD ($\log L = 14027.4$) by the likelihood ratio test, we see no evidence to suggest that the shape parameters, p_1 and p_2, are significantly different. So, the left and right tails of the returns behave similarly.

We see that the best fitting model for each of the five data sets is one of the two recently introduced distributions, the asymmetric Student's t distribution or the asymmetric exponential power distribution. None of the existing or commonly used models for Z_t provide the best fits. Furthermore, for four of the five data sets, the tails of the returns are asymmetric. The tails are symmetric only for Silver.

[Table 8 about here.]

The best fitting models are summarized in Table 8. Also given in this table are p-values for the best fitting models based on the Cramer-von Mises statistic, the Kolmogorov-Smirnov statistic and the Pearson chi-square statistic. These p-values suggest that each best fitting model provides an adequate description of the data. The p-values appear largest for Gold data. They appear second largest for Brent crude oil data. They appear smallest for Cocoa bean data, West Texas intermediate crude oil data and Silver data.

[Tables 9 and 10 about here.]

We now give some measures of goodness of the best fitted models. These measures are obtained by comparing the observed values of mean, standard deviation and value at risk over windows of length w with fitted values. We use two criteria for comparison: mean absolute deviation and mean squared error. Table 9 gives the mean absolute deviations for mean, standard deviation, $\mathrm{VaR}_{0.9}$ and $\mathrm{VaR}_{0.99}$ for $w = 10, 50, 100$ days. Table 10 gives the mean squared errors for mean, standard deviation, $\mathrm{VaR}_{0.9}$ and $\mathrm{VaR}_{0.99}$ for $w = 10, 50, 100$ days. The standard deviation for Cocoa bean does not exist since its best fitting model is the asymmetric Student's t distribution with $\widehat{\nu_2} = 1.827 < 2$.

The mean absolute deviations and the mean squared errors appear small enough to suggest that the best fitting models are reasonable. The mean absolute deviations and the mean squared errors appear smallest for Gold data. They appear largest for Cocoa bean data, West Texas intermediate crude oil data and Silver data. However, there is no evidence to suggest that the mean absolute deviations or the mean squared errors vary significantly with respect to w.

[Figure 2 about here.]

Boxplots of the fitted values of VaR_p, $p = 0.9, 0.95, 0.975, 0.99$ for the five commodities are shown in Figure 2. We can observe the following: the median of value at risk is largest for West Texas intermediate crude oil and smallest for Gold when $p = 0.9$ or $p = 0.95$; the median of value at risk is largest for Cocoa bean and smallest for Gold when $p = 0.975$ or $p = 0.99$; the variability of value at risk is largest for Cocoa bean and smallest for Gold for every p; the variability of value at risk decreases with p for each commodity.

[Figure 3 about here.]

Boxplots of the fitted values of ES_p, $p = 0.9, 0.95, 0.975, 0.99$ for the five commodities are shown in Figure 3. We can observe the following: the median of expected shortfall is largest for Cocoa bean and smallest for Gold for every p; the variability of expected shortfall is largest for Cocoa bean and smallest for Gold for every p; the variability of expected shortfall decreases with p for each commodity.

[Figure 4 about here.]

Figure 4 shows how the estimates of the expected volatility, $\widehat{\omega} + \widehat{\alpha_1} \widehat{E} \left(X_{i-1}^2 \right) + \widehat{\beta}_1 \sigma_{i-1}^2$, vary with respect to time for the best fitting models. We can observe the following: the expected volatility for Brent crude oil and West Texas intermediate crude oil increases monotonically and sharply with respect to time; the expected volatility for Gold and Silver increases monotonically before approaching an asymptote; the expected volatility for all t is largest for Brent crude oil; the expected volatility for small t is second largest for West Texas intermediate crude oil; the expected volatility for all sufficiently large t is second largest for Gold; the expected volatility for small t is third largest for Gold; the expected volatility for all sufficiently large t is third largest for West Texas intermediate crude oil; the expected volatility for small t is smallest for Gold; the expected volatility for all sufficiently large t is smallest for Silver. The expected volatility for Cocoa bean does not exist since its best fitting model is the asymmetric Student's t distribution with $\widehat{\nu_2} = 1.827 < 2$.

[Figure 5 about here.]

Finally, Figure 5 gives forecasts of VaR_p, $p = 0.9, 0.95, 0.975, 0.99$ by one hundred additional days. We can observe the following: the forecasts for each commodity increase monotonically with respect to time; the forecasts for each commodity increase monotonically with respect to p; the forecasts are largest for Silver for every p; the forecasts are second largest for West Texas intermediate crude oil for every p; the forecasts are third largest for Brent crude oil for every p; the forecasts are smallest for Gold for every p.

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	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Min	-1.928×10^{-1}	-1.363×10^{-1}	-1.722×10^{-1}	-7.143×10^{-2}	-1.869×10^{-1}
Q1	-6.094×10^{-3}	-1.097×10^{-2}	-1.147×10^{-2}	-3.856×10^{-3}	-8.609×10^{-3}
Median	0	3.335×10^{-4}	5.863×10^{-4}	0	0
Mean	1.528×10^{-4}	3.352×10^{-4}	2.897×10^{-4}	3.023×10^{-4}	3.985×10^{-4}
Q3	6.431×10^{-3}	1.245×10^{-2}	1.264×10^{-2}	4.791×10^{-3}	9.989×10^{-3}
Max	1.938×10^{-1}	1.35×10^{-1}	2.128×10^{-1}	7.382×10^{-2}	1.828×10^{-1}
SD	1.787×10^{-2}	2.154×10^{-2}	2.379×10^{-2}	1.008×10^{-2}	2.017×10^{-2}
CV	117.007	64.280	41.632	33.346	50.610
Skewness	4.248×10^{-2}	-9.604×10^{-2}	-4.463×10^{-2}	-1.485×10^{-1}	-3.669×10^{-1}
Kurtosis	19.949	6.019	8.411	9.277	12.352
IQR	1.253×10^{-2}	2.342×10^{-2}	2.411×10^{-2}	8.647×10^{-3}	1.860×10^{-2}
Range	3.866×10^{-1}	2.713×10^{-1}	3.85×10^{-1}	1.453×10^{-1}	3.697×10^{-1}

Table 1: Summary statistics of the data on the five commodities.

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
AD test	0	5.708×10^{-69}	0	1.480×10^{-167}	0
CVM test	0	2.690×10^{-87}	7.37×10^{-10}	0	0
KS test	1.070×10^{-245}	1.874×10^{-36}	1.110×10^{-53}	3.436×10^{-119}	1.475×10^{-94}
Pearson test	0	5.603×10^{-82}	3.805×10^{-285}	4.182×10^{-283}	0
JB test	0	0	0	0	0
G test	0	3.770×10^{-138}	8.817×10^{-247}	0	0
DDS test	0	0	0	0	0

Table 2: Tests for normality of the data on the five commodities.

Model	Parameter estimates	$-\log L$	AIC
Gaussian	$\widehat{\alpha_1} = 2.364 \times 10^{-2}$, $\widehat{\beta_1} = 9.590 \times 10^{-1}$, $\widehat{\omega} = 5.154 \times 10^{-6}$, $\widehat{\mu} = 1.396 \times 10^{-2}$	-13924.5	-27841.0
Student's t	$\widehat{\alpha_1} = 1.782 \times 10^{-1}$, $\widehat{\beta_1} = 7.246 \times 10^{-1}$, $\widehat{\omega} = 5.888 \times 10^{-17}$, $\widehat{\mu} = -6.272 \times 10^{-3}$, $\widehat{\nu} = 5.656$	-15133.3	-30256.7
SSTD	$\widehat{\alpha_1} = 1.782 \times 10^{-1}$, $\widehat{\beta_1} = 7.246 \times 10^{-1}$, $\widehat{\omega} = 1.998 \times 10^{-18}$, $\widehat{\mu} = 5.145 \times 10^{-3}$, $\widehat{\nu} = 5.622$, $\widehat{\alpha} = 4.949 \times 10^{-1}$	-15148.2	-30284.3
AST	$\widehat{\alpha_1} = 1.782 \times 10^{-1}$, $\widehat{\beta_1} = 7.246 \times 10^{-1}$, $\widehat{\omega} = 1.998 \times 10^{-18}$, $\widehat{\mu} = 5.145 \times 10^{-3}$, $\widehat{\nu_1} = 5.622$, $\widehat{\nu_2} = 1.827$, $\widehat{\alpha} = 4.949 \times 10^{-1}$	-15479.6	-30945.2
GED	–	–	–
SEPD	–	–	–
AEPD	–	–	–
SNORM	$\widehat{\alpha_1} = 2.528 \times 10^{-2}$, $\widehat{\beta_1} = 9.572 \times 10^{-1}$, $\widehat{\omega} = 5.266 \times 10^{-6}$, $\widehat{\mu} = 2.793 \times 10^{-2}$, $\widehat{\lambda} = 1.039$	-13929.8	-27849.6
SGED	–	–	–
SSTD0	–	–	–
SNIG	–	–	–

Table 3: Fitted models and estimates for Cocoa bean data.

Model	Parameter estimates	$-\log L$	AIC
Gaussian	$\widehat{\alpha_1} = 4.807 \times 10^{-2}$, $\widehat{\beta_1} = 9.461 \times 10^{-1}$, $\widehat{\omega} = 2.693 \times 10^{-6}$, $\widehat{\mu} = 2.004 \times 10^{-2}$	-13004.1	-26000.2
Student's t	$\widehat{\alpha_1} = 4.808 \times 10^{-2}$, $\widehat{\beta_1} = 9.461 \times 10^{-1}$, $\widehat{\omega} = 3.793 \times 10^{-7}$, $\widehat{\mu} = 3.211 \times 10^{-2}$, $\widehat{\nu} = 1.024 \times 10^{1}$	-13077.7	-26145.5
SSTD	$\widehat{\alpha_1} = 4.808 \times 10^{-2}$, $\widehat{\beta_1} = 9.461 \times 10^{-1}$, $\widehat{\omega} = 3.757 \times 10^{-7}$, $\widehat{\mu} = 1.128 \times 10^{-1}$, $\widehat{\nu} = 1.036 \times 10^{1}$, $\widehat{\alpha} = 5.277 \times 10^{-1}$	-13082.2	-26152.5
AST	$\widehat{\alpha_1} = 4.808 \times 10^{-2}$, $\widehat{\beta_1} = 9.461 \times 10^{-1}$, $\widehat{\omega} = 3.736 \times 10^{-7}$, $\widehat{\mu} = 8.857 \times 10^{-2}$, $\widehat{\nu_1} = 9.181$, $\widehat{\nu_2} = 1.197 \times 10^{1}$, $\widehat{\alpha} = 5.187 \times 10^{-1}$	-13082.6	-26151.1
GED	$\widehat{\alpha_1} = 4.491 \times 10^{-2}$, $\widehat{\beta_1} = 9.501 \times 10^{-1}$, $\widehat{\omega} = 2.747 \times 10^{-7}$, $\widehat{\mu} = 2.909 \times 10^{-2}$, $\widehat{a} = 1.501$	-13088.4	-26166.7
SEPD	$\widehat{\alpha_1} = 3.546 \times 10^{-2}$, $\widehat{\beta_1} = 9.560 \times 10^{-1}$, $\widehat{\omega} = 3.777 \times 10^{-7}$, $\widehat{\mu} = 8.507 \times 10^{-2}$, $\widehat{p} = 1.542$, $\widehat{\alpha} = 5.085 \times 10^{-1}$	-13095.5	-26178.9
AEPD	$\widehat{\alpha_1} = 3.546 \times 10^{-2}$, $\widehat{\beta_1} == 9.560 \times 10^{-1}$, $\widehat{\omega} = 4.726 \times 10^{-7}$, $\widehat{\mu} = 3.349 \times 10^{-4}$, $\widehat{p_1} = 1.319$, $\widehat{p_2} = 1.562$, $\widehat{\alpha} = 4.817 \times 10^{-1}$	-13110.5	-26206.9
SNORM	$\widehat{\alpha_1} = 4.652 \times 10^{-2}$, $\widehat{\beta_1} = 9.483 \times 10^{-1}$, $\widehat{\omega} = 2.370 \times 10^{-6}$, $\widehat{\mu} = 1.581 \times 10^{-2}$, $\widehat{\lambda} = 9.449 \times 10^{-1}$	-13010.1	-26010.2
SGED	$\widehat{\alpha_1} = 4.407 \times 10^{-2}$, $\widehat{\beta_1} = 9.511 \times 10^{-1}$, $\widehat{\omega} = 2.094 \times 10^{-6}$, $\widehat{\mu} = 1.668 \times 10^{-2}$, $\widehat{\lambda} = 9.696 \times 10^{-1}$, $\widehat{k} = 1.364$	-13108.1	-26204.2
SSTD0	$\widehat{\alpha_1} = 4.182 \times 10^{-2}$, $\widehat{\beta_1} = 9.542 \times 10^{-1}$, $\widehat{\omega} = 1.684 \times 10^{-6}$, $\widehat{\mu} = 2.170 \times 10^{-2}$, $\widehat{\gamma} = 9.492 \times 10^{-1}$, $\widehat{\nu} = 7.461$	-13097.2	-26182.5
SNIG	$\widehat{\alpha_1} = 4.241 \times 10^{-2}$, $\widehat{\beta_1} = 9.535 \times 10^{-1}$, $\widehat{\omega} = 1.726 \times 10^{-6}$, $\widehat{\mu} = 2.095 \times 10^{-2}$, $\widehat{\alpha} = 2.182$, $\widehat{\beta} = -8.861 \times 10^{-2}$	-13101.5	-26191.0

Table 4: Fitted models and estimates for Brent crude oil data.

Model	Parameter estimates	$-\log L$	AIC
Gaussian	$\widehat{\alpha_1} = 5.952 \times 10^{-2}$, $\widehat{\beta_1} = 9.267 \times 10^{-1}$, $\widehat{\omega} = 7.877 \times 10^{-6}$, $\widehat{\mu} = 1.331 \times 10^{-2}$	-12545.8	-25083.5
Student's t	$\widehat{\alpha_1} = 5.952 \times 10^{-2}$, $\widehat{\beta_1} = 9.266 \times 10^{-1}$, $\widehat{\omega} = 9.503 \times 10^{-7}$, $\widehat{\mu} = 3.176 \times 10^{-2}$ $\widehat{\nu} = 7.051$	-12691.0	-25372.1
SSTD	$\widehat{\alpha_1} = 5.952 \times 10^{-2}$, $\widehat{\beta_1} = 9.266 \times 10^{-1}$, $\widehat{\omega} = 9.881 \times 10^{-7}$, $\widehat{\mu} = 1.020 \times 10^{-1}$, $\widehat{\nu} = 7.138$, $\widehat{\alpha} = 5.250 \times 10^{-1}$	-12694.9	-25377.8
AST	$\widehat{\alpha_1} = 5.952 \times 10^{-2}$, $\widehat{\beta_1} = 9.266 \times 10^{-1}$, $\widehat{\omega} = 9.666 \times 10^{-7}$, $\widehat{\mu} = 6.449 \times 10^{-2}$, $\widehat{\nu_1} = 6.152$, $\widehat{\nu_2} = 8.502$, $\widehat{\alpha} = 5.111 \times 10^{-1}$	-12696.0	-25377.9
GED	$\widehat{\alpha_1} = 4.488 \times 10^{-2}$, $\widehat{\beta_1} = 9.469 \times 10^{-1}$, $\widehat{\omega} = 3.221 \times 10^{-7}$, $\widehat{\mu} = 1.521 \times 10^{-2}$, $\widehat{a} = 1.334$	-12709.2	-25408.4
SEPD	$\widehat{\alpha_1} = 1.784 \times 10^{-2}$, $\widehat{\beta_1} = 9.695 \times 10^{-1}$, $\widehat{\omega} = 1.434 \times 10^{-6}$, $\widehat{\mu} = 1.856 \times 10^{-2}$, $\widehat{p} = 1.313$, $\widehat{\alpha} = 5.018550 \times 10^{-1}$	-12736.8	-25461.6
AEPD	$\widehat{\alpha_1} = 1.784 \times 10^{-2}$, $\widehat{\beta_1} = 9.695 \times 10^{-1}$, $\widehat{\omega} = 7.956 \times 10^{-7}$, $\widehat{\mu} = -1.933 \times 10^{-5}$, $\widehat{p_1} = 1.076$, $\widehat{p_2} = 1.249$, $\widehat{\alpha} = 4.801 \times 10^{-1}$	-12755.1	-25496.2
SNORM	$\widehat{\alpha_1} = 5.895 \times 10^{-2}$, $\widehat{\beta_1} = 9.279 \times 10^{-1}$, $\widehat{\omega} = 7.503 \times 10^{-6}$, $\widehat{\mu} = 6.610 \times 10^{-3}$, $\widehat{\lambda} = 9.435 \times 10^{-1}$	-12553.0	-25096.0
SGED	$\widehat{\alpha_1} = 4.488 \times 10^{-2}$, $\widehat{\beta_1} = 9.468 \times 10^{-1}$, $\widehat{\omega} = 4.655 \times 10^{-6}$, $\widehat{\mu} = 8.583 \times 10^{-3}$, $\widehat{\lambda} = 1.005$, $\widehat{k} = 1.155$	-12747.0	-25482
SSTD0	$\widehat{\alpha_1} = 3.961 \times 10^{-2}$, $\widehat{\beta_1} = 9.541 \times 10^{-1}$, $\widehat{\omega} = 3.451 \times 10^{-6}$, $\widehat{\mu} = 1.650 \times 10^{-2}$, $\widehat{\gamma} == 9.569 \times 10^{-1}$, $\widehat{\nu} = 5.427$	-12728.2	-25444.3
SNIG	–	–	–

Table 5: Fitted models and estimates for West Texas intermediate crude oil data.

Model	Parameter estimates	$-\log L$	AIC
Gaussian	$\widehat{\alpha_1} = 4.525 \times 10^{-2}$, $\widehat{\beta_1} = 9.539 \times 10^{-1}$, $\widehat{\omega} = 2.788 \times 10^{-7}$, $\widehat{\mu} = 3.686 \times 10^{-2}$	-17431.9	-34855.7
Student's t	$\widehat{\alpha_1} = 2.442 \times 10^{-2}$, $\widehat{\beta_1} = 9.539 \times 10^{-1}$, $\widehat{\omega} = 8.334 \times 10^{-8}$, $\widehat{\mu} = 5.885 \times 10^{-2}$, $\widehat{\nu} = 3.821$	-17745.3	-35480.6
SSTD	$\widehat{\alpha_1} = 3.563 \times 10^{-2}$, $\widehat{\beta_1} = 9.343 \times 10^{-1}$, $\widehat{\omega} = 1.382 \times 10^{-7}$, $\widehat{\mu} = 5.479 \times 10^{-2}$, $\widehat{\nu} = 3.939$, $\widehat{\alpha} = 4.990 \times 10^{-1}$	-17745.8	-35479.6
AST	$\widehat{\alpha_1} = 3.516 \times 10^{-2}$, $\widehat{\beta_1} = 9.343 \times 10^{-1}$, $\widehat{\omega} = 1.345 \times 10^{-7}$, $\widehat{\mu} = 5.527 \times 10^{-2}$, $\widehat{\nu_1} = 3.636$, $\widehat{\nu_2} = 3.969$, $\widehat{\alpha} = 4.986 \times 10^{-1}$	-17746.3	-35478.7
GED	$\widehat{\alpha_1} = 2.730 \times 10^{-2}$, $\widehat{\beta_1} = 9.477 \times 10^{-1}$, $\widehat{\omega} = 1.135 \times 10^{-7}$, $\widehat{\mu} = 1.652 \times 10^{-6}$, $\widehat{a} = 1.006$	-17781.6	-35553.2
SEPD	$\widehat{\alpha_1} = 2.723 \times 10^{-2}$, $\widehat{\beta_1} = 9.477 \times 10^{-1}$, $\widehat{\omega} = 1.132 \times 10^{-7}$, $\widehat{\mu} = -6.267 \times 10^{-8}$, $\widehat{p} = 1.005$, $\widehat{\alpha} = 4.869 \times 10^{-1}$	-17785.2	-35558.3
AEPD	$\widehat{\alpha_1} = 2.714 \times 10^{-2}$, $\widehat{\beta_1} = 9.477 \times 10^{-1}$, $\widehat{\omega} = 1.179 \times 10^{-7}$, $\widehat{\mu} = -9.930 \times 10^{-8}$, $\widehat{p_1} = 9.672 \times 10^{-1}$, $\widehat{p_2} = 1.042$, $\widehat{\alpha} = 4.777 \times 10^{-1}$	-17787.7	-35561.4
SNORM	$\widehat{\alpha_1} = 4.526 \times 10^{-2}$, $\widehat{\beta_1} = 9.539 \times 10^{-1}$, $\widehat{\omega} = 2.766 \times 10^{-7}$, $\widehat{\mu} = 3.761 \times 10^{-2}$, $\widehat{\lambda} = 1.004$	-17431.9	-34853.8
SGED	$\widehat{\alpha_1} = 5.412 \times 10^{-2}$, $\widehat{\beta_1} = 9.478 \times 10^{-1}$, $\widehat{\omega} = 2.252 \times 10^{-7}$, $\widehat{\mu} = 3.687 \times 10^{-2}$, $\widehat{\lambda} = 1.026$, $\widehat{k} = 1.005$	-17785.1	-35558.3
SSTD0	$\widehat{\alpha_1} = 5.929 \times 10^{-2}$, $\widehat{\beta_1} = 9.469 \times 10^{-1}$, $\widehat{\omega} = 2.091 \times 10^{-7}$, $\widehat{\mu} = 3.636 \times 10^{-2}$, $\widehat{\gamma} = 9.904 \times 10^{-1}$, $\widehat{\nu} = 3.829$	-17747.2	-35482.4
SNIG	–	–	–

Table 6: Fitted models and estimates for Gold data.

Model	Parameter estimates	$-\log L$	AIC
Gaussian	$\widehat{\alpha_1} = 6.133 \times 10^{-2}$, $\widehat{\beta_1} = 9.369 \times 10^{-1}$, $\widehat{\omega} = 1.670 \times 10^{-6}$, $\widehat{\mu} = 2.694 \times 10^{-2}$	-13751.4	-27494.8
Student's t	$\widehat{\alpha_1} = 6.133 \times 10^{-2}$, $\widehat{\beta_1} = 9.369 \times 10^{-1}$, $\widehat{\omega} = 8.497 \times 10^{-8}$, $\widehat{\mu} = 2.800 \times 10^{-2}$, $\widehat{\nu} = 8.413$	-13912.2	-27814.3
SSTD	$\widehat{\alpha_1} = 6.133 \times 10^{-2}$, $\widehat{\beta_1} = 9.369 \times 10^{-1}$, $\widehat{\omega} = 8.527 \times 10^{-8}$, $\widehat{\mu} = 2.745 \times 10^{-2}$, $\widehat{\nu} = 8.417$, $\widehat{\alpha} = 4.998 \times 10^{-1}$	-13912.2	
AST	$\widehat{\alpha_1} = 6.133 \times 10^{-2}$, $\widehat{\beta_1} = 9.369 \times 10^{-1}$, $\widehat{\omega} = 8.454 \times 10^{-8}$, $\widehat{\mu} = 4.933 \times 10^{-2}$, $\widehat{\nu_1} = 9.416$, $\widehat{\nu_2} = 7.644$, $\widehat{\alpha} = 5.083 \times 10^{-1}$	-13912.5	-27811
GED	$\widehat{\alpha_1} = 2.809 \times 10^{-2}$, $\widehat{\beta_1} = 9.469 \times 10^{-1}$, $\widehat{\omega} = 8.624 \times 10^{-7}$, $\widehat{\mu} = 7.075 \times 10^{-7}$, $\widehat{a} = 1.065$	-14025.5	-28041
SEPD	$\widehat{\alpha_1} = 2.806 \times 10^{-2}$, $\widehat{\beta_1} = 9.469 \times 10^{-1}$, $\widehat{\omega} = 8.588 \times 10^{-7}$, $\widehat{\mu} = -4.391 \times 10^{-7}$, $\widehat{p} = 1.065$, $\widehat{\alpha} = 4.905 \times 10^{-1}$	-14027.4	-28042.8
AEPD	$\widehat{\alpha_1} = 2.796 \times 10^{-2}$, $\widehat{\beta_1} = 9.469 \times 10^{-1}$, $\widehat{\omega} = 8.743 \times 10^{-7}$, $\widehat{\mu} = 5.434 \times 10^{-9}$, $\widehat{p_1} = 1.046$, $\widehat{p_2} = 1.079$, $\widehat{\alpha} = 4.866 \times 10^{-1}$	-14027.8	-28041.7
SNORM	$\widehat{\alpha_1} = 1.004$, $\widehat{\beta_1} = 6.139 \times 10^{-2}$, $\widehat{\omega} = 9.369 \times 10^{-1}$, $\widehat{\mu} = 1.664 \times 10^{-6}$, $\widehat{\lambda} = 2.746 \times 10^{-2}$	-13751.4	-27492.8
SGED	$\widehat{\alpha_1} = 5.170 \times 10^{-2}$, $\widehat{\beta_1} = 9.469 \times 10^{-1}$, $\widehat{\omega} = 1.581 \times 10^{-6}$, $\widehat{\mu} = 2.718 \times 10^{-2}$, $\widehat{\lambda} = 1.019$, $\widehat{k} = 1.065$	-14027.4	-28042.8
SSTD0	$\widehat{\alpha_1} = 4.927 \times 10^{-2}$, $\widehat{\beta_1} = 9.497 \times 10^{-1}$, $\widehat{\omega} = 1.708 \times 10^{-6}$, $\widehat{\mu} = 2.801 \times 10^{-2}$, $\widehat{\gamma} == 1.002$, $\widehat{\nu} = 4.372$	-14002.6	-27993.3
SNIG	$\widehat{\alpha_1} = 5.041 \times 10^{-2}$, $\widehat{\beta_1} = 9.486 \times 10^{-1}$, $\widehat{\omega} = 8.784 \times 10^{-7}$, $\widehat{\mu} = 2.633 \times 10^{-2}$, $\widehat{\alpha} = 1.042$, $\widehat{\beta} = -4.255 \times 10^{-3}$	-14005.9	-27999.9

Table 7: Fitted models and estimates for Silver data.

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Best model	AST	AEPD	AEPD	AEPD	SEPD
CVM test p-value	0.061	0.094	0.064	0.134	0.060
KS test p-value	0.059	0.089	0.061	0.223	0.051
Pearson test p-value	0.052	0.088	0.066	0.185	0.063

Table 8: Best fitting models.

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Mean ($w = 10$)	4.938×10^{-3}	5.301×10^{-3}	5.453×10^{-3}	2.263×10^{-3}	4.192×10^{-3}
SD ($w = 10$)	–	2.148×10^{-4}	2.998×10^{-4}	4.842×10^{-5}	2.249×10^{-4}
$\text{VaR}_{0.9}$ ($w = 10$)	1.107×10^{-2}	8.582×10^{-3}	9.510×10^{-3}	4.030×10^{-3}	8.240×10^{-3}
$\text{VaR}_{0.99}$ ($w = 10$)	6.960×10^{-2}	2.221×10^{-2}	2.757×10^{-2}	1.323×10^{-2}	2.659×10^{-2}
Mean ($w = 50$)	3.578×10^{-3}	2.350×10^{-3}	2.264×10^{-3}	9.679×10^{-4}	1.758×10^{-3}
SD ($w = 50$)	–	1.168×10^{-4}	1.558×10^{-4}	2.664×10^{-5}	1.520×10^{-4}
$\text{VaR}_{0.9}$ ($w = 50$)	9.125×10^{-3}	4.239×10^{-3}	4.636×10^{-3}	2.162×10^{-3}	4.804×10^{-3}
$\text{VaR}_{0.99}$ ($w = 50$)	5.701×10^{-2}	1.011×10^{-2}	1.436×10^{-2}	7.968×10^{-3}	1.647×10^{-2}
Mean ($w = 100$)	3.450×10^{-3}	1.566×10^{-3}	1.620×10^{-3}	7.047×10^{-4}	1.156×10^{-3}
SD ($w = 100$)	–	1.276×10^{-4}	1.408×10^{-4}	3.206×10^{-5}	1.465×10^{-4}
$\text{VaR}_{0.9}$ ($w = 100$)	8.968×10^{-3}	4.072×10^{-3}	4.309×10^{-3}	1.993×10^{-3}	4.397×10^{-3}
$\text{VaR}_{0.99}$ ($w = 100$)	5.439×10^{-2}	9.738×10^{-3}	1.278×10^{-2}	7.253×10^{-3}	1.374×10^{-2}

Table 9: Mean absolution deviations as measures of goodness of the best fitting models.

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Mean ($w = 10$)	4.431×10^{-5}	4.759×10^{-5}	5.172×10^{-5}	9.232×10^{-6}	3.393×10^{-5}
SD ($w = 10$)	–	1.119×10^{-7}	2.909×10^{-7}	8.670×10^{-9}	3.513×10^{-7}
$\mathrm{VaR}_{0.9}$ ($w = 10$)	2.711×10^{-4}	1.278×10^{-4}	1.573×10^{-4}	2.957×10^{-5}	1.309×10^{-4}
$\mathrm{VaR}_{0.99}$ ($w = 10$)	7.435×10^{-3}	6.723×10^{-4}	9.940×10^{-4}	2.456×10^{-4}	1.001×10^{-3}
Mean ($w = 50$)	2.027×10^{-5}	1.041×10^{-5}	9.818×10^{-6}	1.552×10^{-6}	5.380×10^{-6}
SD ($w = 50$)	–	3.127×10^{-8}	6.773×10^{-8}	2.226×10^{-9}	1.104×10^{-7}
$\mathrm{VaR}_{0.9}$ ($w = 50$)	4.239×10^{-3}	3.409×10^{-5}	3.855×10^{-5}	8.698×10^{-6}	5.053×10^{-5}
$\mathrm{VaR}_{0.99}$ ($w = 50$)	1.011×10^{-2}	1.572×10^{-4}	3.233×10^{-4}	1.011×10^{-4}	4.454×10^{-4}
Mean ($w = 100$)	1.875×10^{-5}	5.754×10^{-6}	5.493×10^{-6}	7.764×10^{-7}	2.875×10^{-6}
SD ($w = 100$)	–	4.214×10^{-8}	6.420×10^{-8}	3.385×10^{-9}	8.256×10^{-8}
$\mathrm{VaR}_{0.9}$ ($w = 100$)	2.229×10^{-4}	3.356×10^{-5}	3.386×10^{-5}	8.056×10^{-6}	4.515×10^{-5}
$\mathrm{VaR}_{0.99}$ ($w = 100$)	5.409×10^{-3}	1.478×10^{-4}	2.679×10^{-4}	8.758×10^{-5}	3.261×10^{-4}

Table 10: Mean squared errors as measures of goodness of the best fitting models.

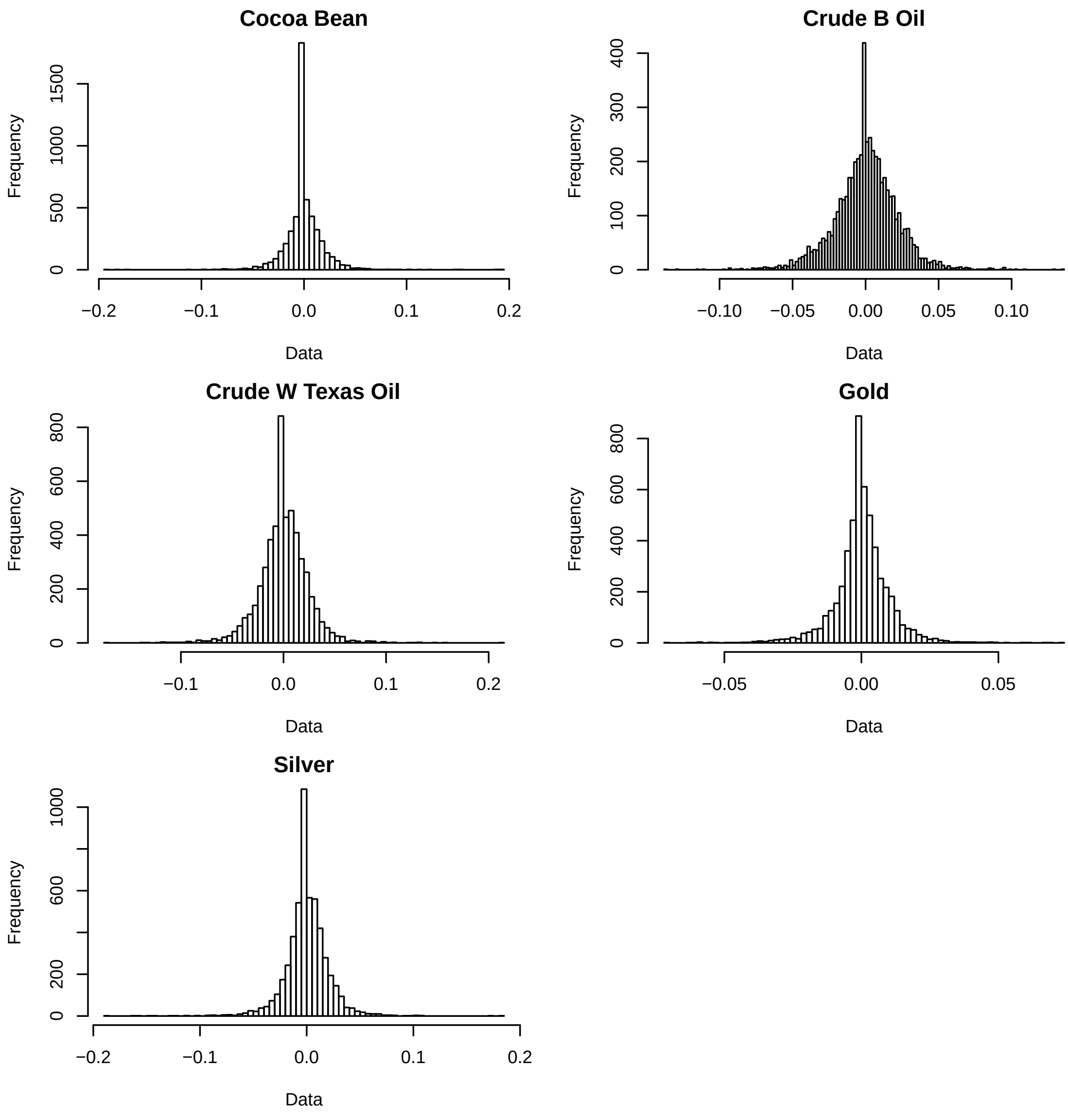

Figure 1: Histogram of the five data sets.

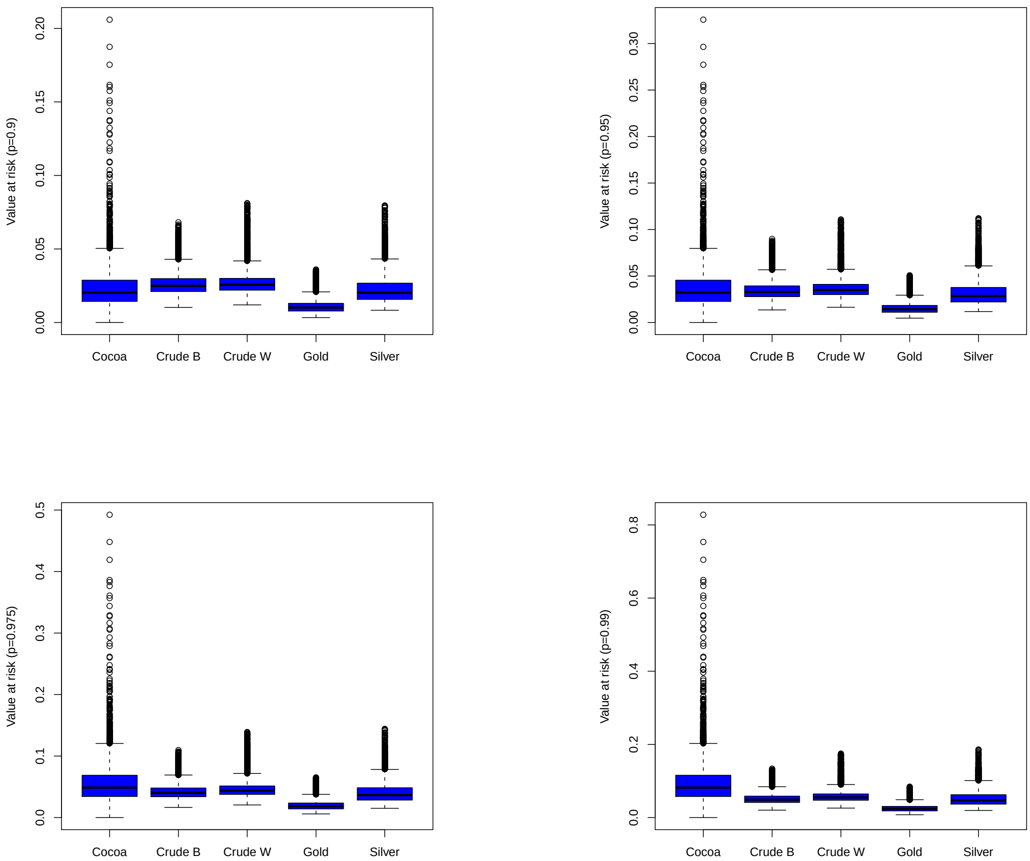

Figure 2: Boxplots of $\mathrm{VaR}_{0.9}$, $\mathrm{VaR}_{0.95}$, $\mathrm{VaR}_{0.975}$ and $\mathrm{VaR}_{0.99}$ for Cocoa bean, Brent crude oil, West Texas intermediate crude oil, Gold and Silver.

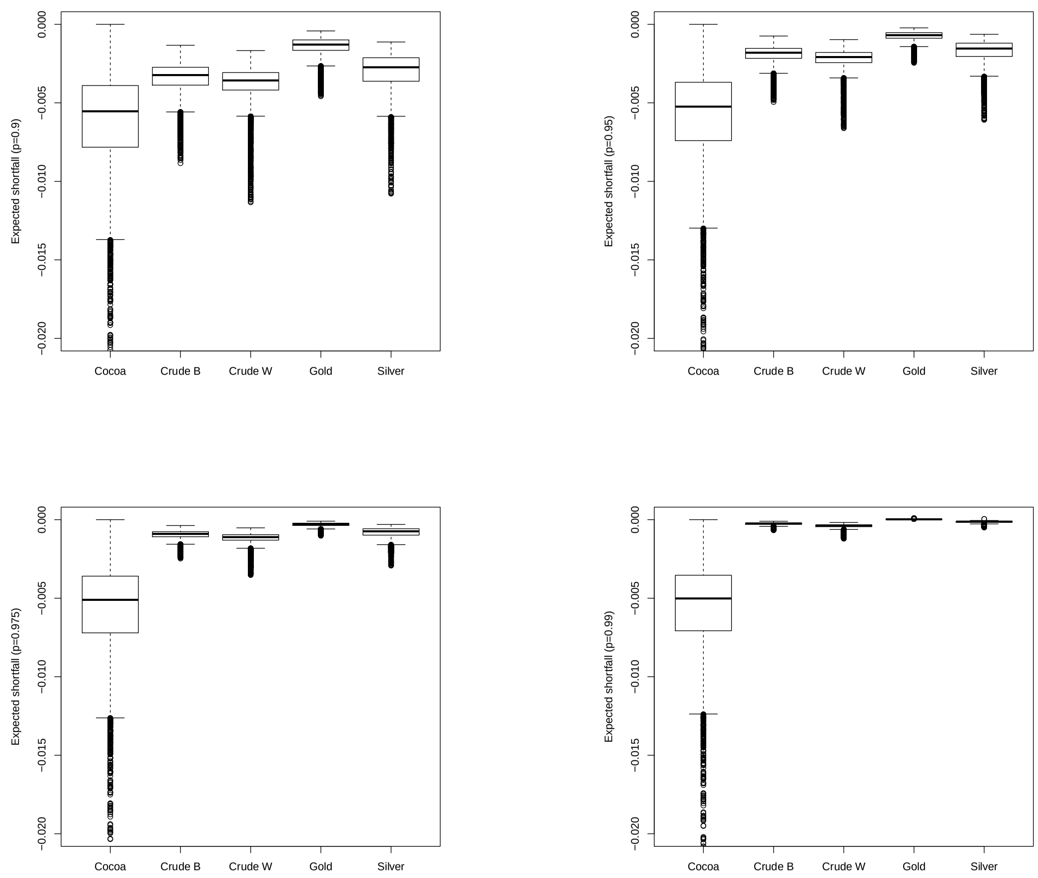

Figure 3: Boxplots of $\mathrm{ES}_{0.9}$, $\mathrm{ES}_{0.95}$, $\mathrm{ES}_{0.975}$ and $\mathrm{ES}_{0.99}$ for Cocoa bean, Brent crude oil, West Texas intermediate crude oil, Gold and Silver.

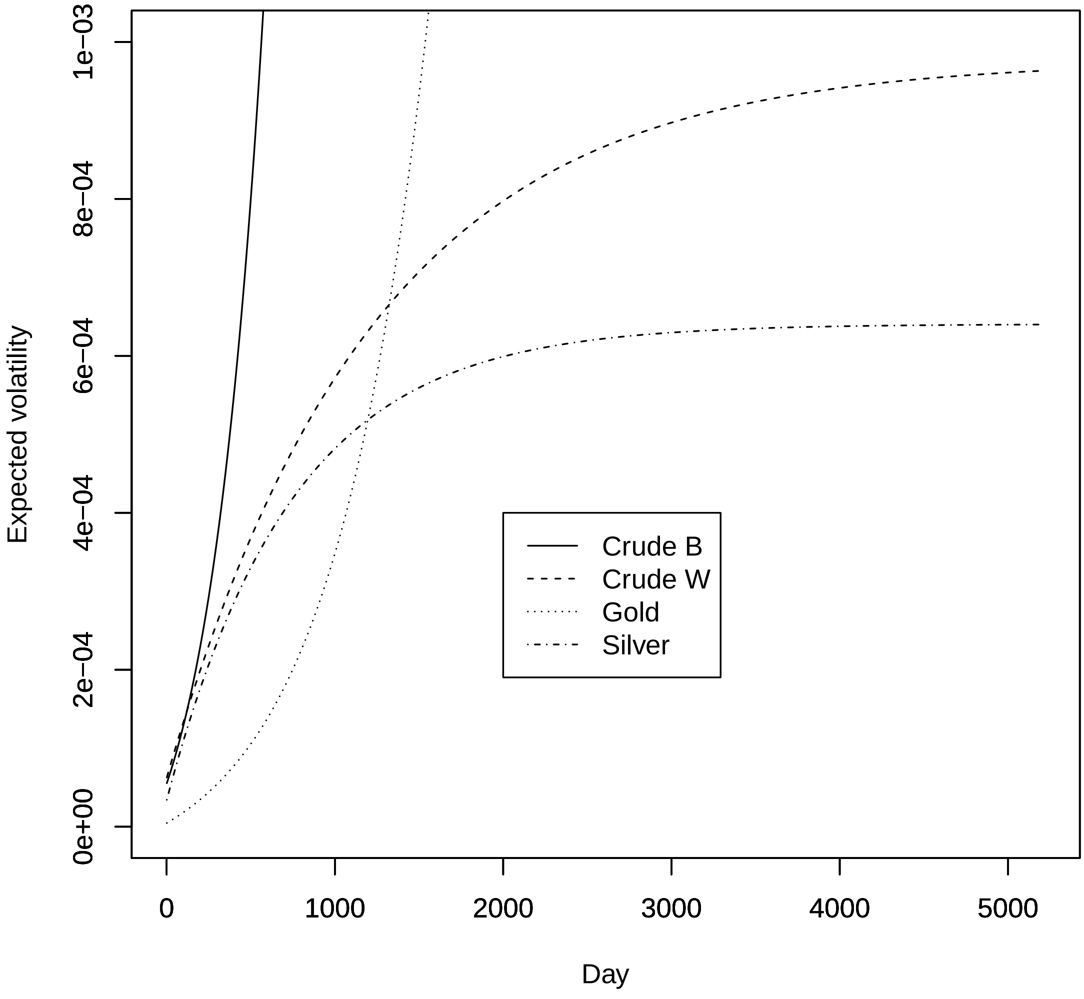

Figure 4: Expected volatility versus time for Brent crude oil, West Texas intermediate crude oil, Gold and Silver.

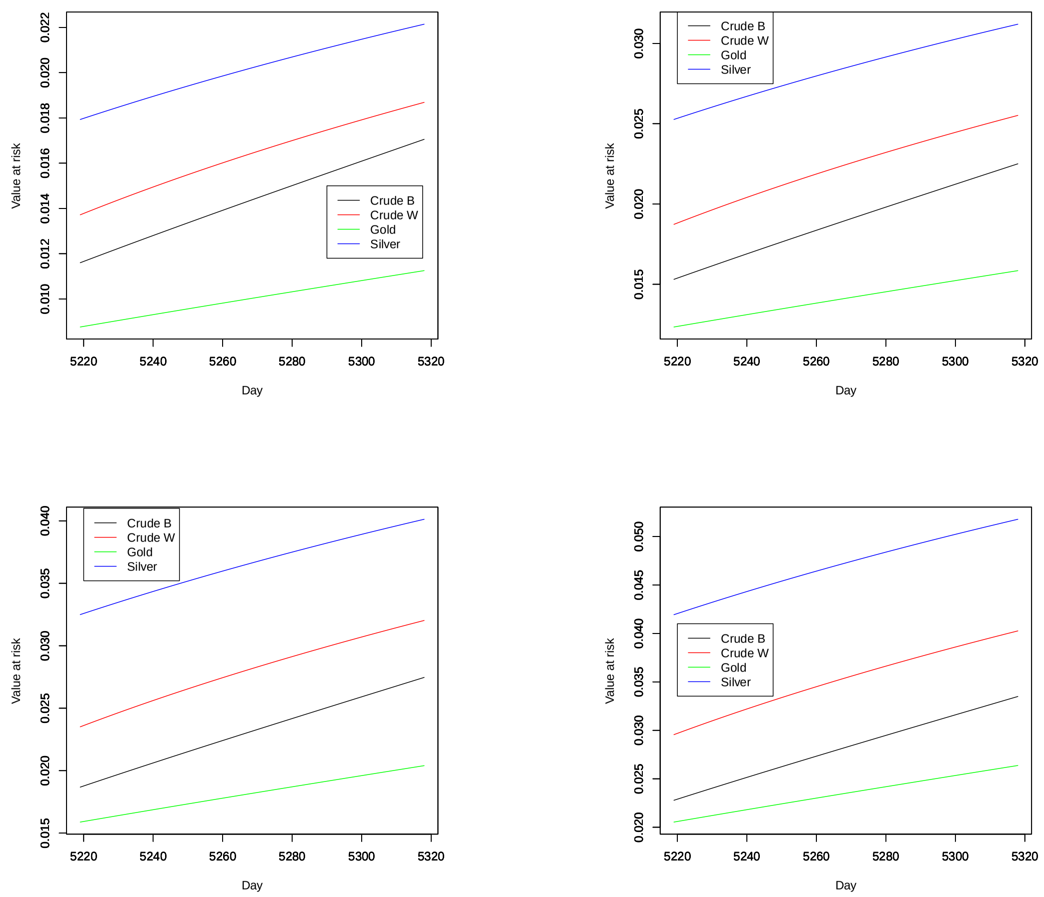

Figure 5: Forecasts of $\mathrm{VaR}_{0.9}$, $\mathrm{VaR}_{0.95}$, $\mathrm{VaR}_{0.975}$ and $\mathrm{VaR}_{0.99}$ for Brent crude oil, West Texas intermediate crude oil, Gold and Silver.