Chapter 12
Bivariate Extreme-Value Distributions

12.1 Preliminaries

The univariate extreme-value distributions consist of types 1 (Gumbel), 2 (Fréchet), and 3. The three types can be transformed to each other. The type 3 distribution of $(-X)$ is the usual Weibull distribution.

In the bivariate context, marginals are of secondary interest compared with the dependence structure. Tiago de Oliveira (1962/63, 1975a,b, 1980, 1984), Gumbel and Goldstein (1964), Gumbel (1965), Gumbel and Mustafi (1967), and Galambos (1987, Chapter 5, especially Section 5.4) assumed Gumbel marginals, whereas de Haan and Resnick (1977) and Kotz and Nadarajah (2000, Chapter 3) chose Fréchet marginals $F(x) = \exp(-x^{-1})$. All three types can be easily transformed to exponential variates, and in most cases we will follow Pickands (1981), Deheuvels (1983, 1985), Smith (1994), and Tawn (1988a) in choosing exponential marginals.

There are several excellent treatises on bivariate and multivariate extreme value distributions; see, for example, Galambos (1987), Smith (1990, 1994), Kotz and Nadarajah (2000), Coles (2001), and Beirlant et al. (2004).

In Section 12.2, we first introduce the bivariate extreme-value distribution. Next, in Section 12.4, we discuss the classical bivariate extreme-value distribution with Gumbel marginals and its properties. Then, in Sections 12.5–12.7, we discuss the bivariate extreme-value distributions with exponential, Fréchet, and Weibull marginal distributions, respectively. In Section 12.8, we describe the methods of derivation, estimation methods are detailed in Section 12.9, and some references to illustrations are presented in Section 12.10. Section 12.11 describes algorithms for the simulation of random variates from the bivariate extreme-value distribution. Some applications are indicated in Section 12.12 and finally conditionally specified bivariate Gumbel distributions are mentioned in Section 12.13.

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12.2 Introduction to Bivariate Extreme-Value Distribution

12.2.1 Definition

Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be n pairs of independent bivariate random variables with $X_{\max} = \max(X_1, \ldots, X_n)$ and $Y_{\max} = \max(Y_1, \ldots, Y_n)$. It is possible to find linear transformations $X_{(n)} = a_n X_{\max} + b_n$ $(a_n > 0)$ and $Y_{(n)} = c_n Y_{\max} + d_n$ $(c_n > 0)$ such that $X_{(n)}$ (and $Y_{(n)}$) is one of the three types of extreme-value distributions as $n \to \infty$. Then, the limiting joint distribution of $X_{(n)}$ and $Y_{(n)}$ is a bivariate extreme-value distribution.

A general definition of a bivariate extreme-value distribution can be presented through a copula [Pickands (1981)]. Let (X, Y) have a joint bivariate extreme-value distribution with marginals $F(x)$ and $G(y)$; then, the associated copula is given by

$$\begin{aligned} C(u,v) &= \Pr\{F(X) \le u, G(Y) \le v\} \\ &= \exp[\log(uv) A\{\log(u)/\log(uv)\}] \end{aligned} \tag{12.1}$$

for all $0 \le u, v \le 1$ in terms of a convex function A defined on $[0,1]$ in such a way that $\max(t, 1-t) \le A(t) \le 1$ for all $0 \le t \le 1$. A is known as the *dependence function*, and we will discuss its properties in Section 12.5.2.

12.2.2 General Properties

- Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be a random sample from a bivariate population with a joint distribution whose copula is C. Let $X_{(n)} = \max\{X_i\}$ and $Y_{(n)} = \max\{Y_i\}$. Then the copula that corresponds to $X_{(n)}$ and $Y_{(n)}$ is

$$C_{(n)}(u,v) = C^n(u^{\frac{1}{n}}, v^{\frac{1}{n}}).$$

  A copula C_* is an extreme-value copula if there exists a copula C such that

$$C_*(u,v) = \lim_{n \to \infty} C^n(u^{\frac{1}{n}} v^{\frac{1}{n}});$$

  see Nelsen (2006, p. 97).
- Shi (2003) has considered a transformation of variables from the copula above with $S = -\log(UV) A\left(\frac{\log U}{\log(UV)}\right)$, $T = \frac{\log U}{\log(UV)}$. It has been shown that S and T are "essentially" independent; this leads to some stochastic representation for the bivariate extreme-value distribution.

- In many bivariate distributions (such as the bivariate normal), $X_{\max}$ and $Y_{\max}$ may be asymptotically independent (as the sample size tends to infinity) even if X and Y are correlated. This is so if $\bar{H}(xy)/\{1-H(x,y)\} \to 0$ as $x, y \to \infty$. This result is due to Geffroy (1958/59).
- Let (X, Y) have a bivariate extreme-value distribution. Then, X and Y are PQD.
- Let $H_1(x,y)$ and $H_2(x,y)$ be two bivariate extreme-value distributions, so their weighted geometric mean is

$$[H_1(x,y)]^{\beta}[H_2(x,y)]^{1-\beta}, \quad 0 \le \beta \le 1,$$

see Gumbel and Goldstein (1964).

12.3 Bivariate Extreme-Value Distributions in General Forms

Gumbel (1958, 1965) has described two general forms for bivariate extreme-value distributions in terms of the marginals (univariate extreme-value distributions):

1. Type A

$$H(x,y) = F(x)G(y)\exp\left\{-\theta\left[\frac{1}{\log F(x)} + \frac{1}{\log G(y)}\right]^{-1}\right\}, \quad 1 \le \theta < 1.$$

The corresponding copula is

$$C(u,v) = uv\exp\left(\frac{-\theta}{\log uv}(\log u \log v)\right).$$

2. Type B

$$H(x,y) = \exp\left\{-\left[(-\log F(x))^m + (-\log G(y))^m\right]^{1/m}\right\}, \quad m \ge 1.$$

The copula that corresponds to the type B extreme-value distribution is

$$C(u,v) = \exp\left(-\left[(-\log u)^m + (-\log v)^m\right]^{1/m}\right).$$

It is an extreme-value copula since $C(u^k, v^k) = C^k(u,v)$; in fact, it is the only Archimedean copula that is also an extreme-value copula, as remarked in Example 1.8. It is called the Gumbel–Hougaard copula in Section 2.6.

The type A bivariate extreme-value distribution is known by some as the (Gumbel) mixed model [see, for example, Yue et al. (2000)], whereas the type B bivariate extreme-value distribution is known as the logistic model.

Restricting to the case where both marginals are Gumbel, Yue and Wang (2004) compared these two models by Monte Carlo experiments. Their results indicate that within the range of $0 \leq \rho \leq 2/3$, both models provide the same joint probabilities and joint return periods, and both may be useful for representing statistical properties of X and Y. When $\rho > 2/3$, only the logistic (type B) model can be applied to the joint distribution of X and Y.

12.4 Classical Bivariate Extreme-Value Distributions with Gumbel Marginals

Three special types are considered in this section—type A, type B, and type C—all having Gumbel marginals. The distributions with exponential marginals will be discussed in Section 12.5.

A bivariate extreme-value distribution with Gumbel marginals has the general form

$$H(x,y) = \exp\left\{-\int_0^1 \min[f_1(s)e^{-x}, f_2(s)e^{-y}]ds\right\}, \tag{12.2}$$

where $f_1(t)$ and $f_2(t)$ are non-negative Lebesgue integrable functions such that $\int_0^1 f_i(t)dt = 1, i = 1, 2$; see, for example, Resnick (1987, p. 272).

12.4.1 Type A Distributions

These distributions are also known as the mixed model.

Formula of the Cumulative Distribution Function

The joint distribution function is

$$H(x,y) = \exp[-e^{-x} - e^{-y} + \theta(e^x + e^y)^{-1}], \quad \theta \leq 1, \tag{12.3}$$

which is an increasing function of θ.

Formula of the Joint Density

The joint density function is

$$\begin{aligned} h(x,y) = e^{-(x+y)}[1 - \theta(e^{2x} + e^{2y})(e^x + e^y)^{-2} + 2\theta e^{2(x+y)}(e^x + e^y)^{-3} \\ +\theta^2 e^{2(x+y)}(e^x + e^y)^{-4}]\exp[-e^{-x} - e^{-y} + \theta(e^x + e^y)^{-1}]. \end{aligned} \tag{12.4}$$

Univariate Properties

The marginal distribution function of X is $F(x) = \exp[-e^{-x}]$, $-\infty < x < \infty$, and a similar expression for $G(y)$. That is, the marginals are both type I extreme-value distributions. Note that the type I extreme-value distribution is also known as the Gumbel distribution. In fact, it is the distribution most commonly referred to in discussions of univariate extreme-value distributions.

Medians and Modes

The median of the common distribution of X and Y is

$$\mu = -\log(\log 2) = 0.36651, \tag{12.5}$$

so that $F(\mu)G(\mu) = \frac{1}{4}$ and

$$H(\mu, \mu) = \exp\left(-2e^{-\mu} + \frac{1}{2}\theta e^{-\mu}\right) = \left(\frac{1}{4}\right)^{1-\theta/4}. \tag{12.6}$$

Also,

$$H(0,0) = (e^{-2})^{1-\theta/4}. \tag{12.7}$$

The value $\tilde{\mu}$, such that $H(\tilde{\mu}, \tilde{\mu}) = \frac{1}{4}$, satisfies the equation

$$\left(2 - \frac{1}{2}\theta\right) e^{-\tilde{\mu}} = 2\log 2, \tag{12.8}$$

and so

$$\tilde{\mu} = \log\left(1 - \frac{1}{4}\theta\right) - \log(\log 2) = \log\left(1 - \frac{1}{4}\theta\right) + 0.3665. \tag{12.9}$$

Since $0 \le \theta \le 1$, $0.3665 - \log(\frac{4}{3}) = 0.0787 \le \tilde{\mu} \le 0.3665$.

The mode of the common distribution of X and Y is at zero. The mode of the joint distribution is at

$$x = y = \log\left[\frac{(2-\theta)(4-\theta)}{2\theta}\left\{\sqrt{\frac{1}{2} + \frac{2}{(2-\theta)^2}} - 1\right\}\right]. \tag{12.10}$$

The numerical values are tabulated, for example, in Table 53.1 of Kotz et al. (2000, p. 627).

Correlation Coefficients

The expression for the product-moment correlation is quite complex. However, Spearman's rho (the grade correlation) is simpler, and is given by

$$\rho_S = 3\left(2-\frac{1}{4}\theta\right)^{-1} \times \left[1+2\left(2\theta-\frac{1}{4}\theta^2\right)^{-1}\tan^{-1}\left\{\left(2\theta-\frac{1}{4}\theta^2\right)^{1/2}\left(2-\frac{1}{2}\theta\right)^{-1}\right\}\right]-3. \quad (12.11)$$

There appears to be a misprint in the formula given by Gumbel and Mustafi (1967, p. 583). However, their Table 3 appears to be correct. Some values of Spearman's rho for a few values of θ can also be found in the same table.

12.4.2 Type B Distributions

Type B bivariate extreme-value distributions are also known as logistic models.

Formula of the Cumulative Distribution Function

The joint cumulative distribution function is

$$H(x,y) = \exp\left[-(e^{-mx}+e^{-my})^{1/m}\right], \quad m \geq 1. \quad (12.12)$$

Since $\lim_{m\to\infty}(e^{-mx}+e^{-my})^{1/m} = \max(e^{-x}, e^{-y})$, we obtain

$$\lim_{m\to\infty} H(x,y) = \min\left[\exp(-e^{-x}), \exp(-e^{-y})\right] = \min\left(F(x), G(y)\right). \quad (12.13)$$

It is clear that, for $m=1$, X and Y are independent.

Formula of the Joint Density

The joint density function is

$$h(x,y) = e^{-m(x+y)}(e^{-mx} + e^{-my})^{-2+1/m}]$$
$$\times\{m - 1 + (e^{-mx} + e^{-my})^{1/m}\}$$
$$\times \exp[-(e^{-mx} + e^{-my})^{1/m}], \quad (12.14)$$

for $m \geq 1$.

Univariate Properties

The marginal distributions are both type I extreme-value distributions.

Medians and Modes

With the univariate median μ defined as $F(\mu) = G(\mu) = \frac{1}{2}$, we find, for type B distributions,

$$H(\mu,\mu) = \left(\frac{1}{4}\right)^{1/m} \quad (12.15)$$

and

$$H(0,0) = (e^{-2})^{1/m} \quad (12.16)$$

[compare these with (12.6) and (12.7)].

The values of $\tilde{\mu}$ such that $H(\tilde{\mu},\tilde{\mu}) = \frac{1}{4}$ satisfies the equation

$$\exp[-2^{1/m}e^{-\tilde{\mu}}] = \frac{1}{4},$$

and so

$$\tilde{\mu} = -\log(\log 2) - \frac{m-1}{m}\log 2. \quad (12.17)$$

Since $m \geq 1$, $0.3665 - \log 2 = -0.3266 \leq \tilde{\mu} \leq 0.3665$.

The mode of the joint distribution is at

$$x = y = (1 + m^{-1})\log 2 - \log\left[\sqrt{(m-1)^2 + 4} - m + 3\right]. \quad (12.18)$$

Some numerical values have been presented in Table 53.1 of Kotz et al. (2000).

Correlation Coefficients

The Pearson product-moment correlation coefficient is $\rho = 1 - m^{-2}$.

Other Properties

The expression $X - Y$ has a logistic distribution with

$$\Pr(X - Y \le t) = (1 + e^{-mt})^{-1}. \tag{12.19}$$

Fisher Information Matrix

Shi (1995b) has derived the Fisher information matrix for the multivariate version of the logistic model.

Type B Bivariate Extreme-Value Distribution with Mixed Gumbel Marginals

Escalante-Sandoval (1998) considered a type B bivariate extreme value distribution (12.12) but with the marginals being the mixtures of Gumbel distributions. The joint distribution was found to be useful for performing flood frequency analysis.

12.4.3 Type C Distributions

For these distributions (also known as the biextremal model), the joint distribution function is

$$H(x, y) = \exp\left[-\max\{e^{-x} + (1-\phi)e^{-y}, e^{-y}\}\right], \quad 0 < \phi < 1. \tag{12.20}$$

The distribution in (12.20) can be generated as the joint distribution of X and

$$Y = \max(X + \log\phi, Z + \log(1-\phi)),$$

where X and Z are mutually independent variables with each having a Gumbel distribution.

The distribution has a singular component along the line $y = x + \log\phi$ since

$$\Pr[Y = X + \log\phi] = \Pr[Z - X \le \log\{\phi/(1-\phi)\}] = \phi. \tag{12.21}$$

Correlation Coefficients

The correlation coefficient is given by

$$\mathrm{corr}(X, Y) = \rho = -6\pi^2 \int_0^{\phi} (1-t)^{-1} \log t \, dt,$$

and the Spearman correlation is $\rho_S = 3\phi/(2+\phi)$.

Medians and Modes

$$H(\mu, \mu) = \frac{1}{4}(2^{\phi}) \tag{12.22}$$

and

$$H(0,0) = (e^{-2})^{1-\phi/2}. \tag{12.23}$$

The value $\tilde{\mu}$, such that $H(\tilde{\mu}, \tilde{\mu}) = \frac{1}{4}$, is given by

$$\tilde{\mu} = -\log\left(\frac{\log 2}{1 - \frac{1}{2}\phi}\right). \tag{12.24}$$

12.4.4 Representations of Bivariate Extreme-Value Distributions with Gumbel Marginals

Tiago de Oliveira (1961) showed that a bivariate distribution with standard type I extreme-value marginals can be defined by a cumulative distribution function of the form

$$H(x, y) = \exp\left\{-(e^{-x} + e^{-y})k(y-x)\right\}, \tag{12.25}$$

where $k(\cdot)$ satisfies the conditions

$$\begin{aligned}
&\lim_{t\to\pm\infty} k(t) = 1,\\
&\frac{d}{dt}\{(1+e^{-t})k(t)\} \le 0,\\
&\frac{d}{dt}\{(1+e^{t})k(t)\} \ge 0,\\
&(1+e^{-t})k''(t) + (1-e^{-t})k'(t) \ge 0.
\end{aligned}$$

Type A is obtained by taking

$$k(t) = 1 - \frac{1}{4}\theta \,\mathrm{sech}^2 \frac{1}{2} t. \tag{12.26}$$

Type B is obtained by taking

$$k(t) = (e^{mt} + 1)^{1/m}(e^t + 1)^{-1}. \tag{12.27}$$

Type C is obtained by taking

$$k(t) = (e^t + 1)^{-1}\{1 - \phi + \max(e^t, \phi)\}, \quad 0 < \phi < 1. \tag{12.28}$$

12.5 Bivariate Extreme-Value Distributions with Exponential Marginals

Pickands (1981) [see also Tawn (1988a)] showed a bivariate extreme-value distribution with unit exponential marginals can be expressed via a dependence function.

12.5.1 Pickands' Dependence Function

Here,

$$\bar{H}(x, y) = \exp\left[-(x + y)A\left(\frac{y}{x + y}\right)\right], \quad x, y > 0, \tag{12.29}$$

where

$$A(w) = \int_0^1 \max[(1 - w)q, w(1 - q)]\frac{dB}{dq}dq, \tag{12.30}$$

in which B is a positive function on $[0, 1]$. In order to have unit exponential marginals, we need

$$1 = \int_0^1 q\frac{dB}{dq}dq = \int_0^1 (1 - q)\frac{dB}{dq}. \tag{12.31}$$

[To deduce this, we successively set $x = 0$ and $y = 0$ in (12.29). We then find that $A(0)$ and $A(1)$ must both be 1 and put these values into (12.31).] It follows from (12.31) that $\frac{1}{2}B$ is the distribution function of a random variable with mean $\frac{1}{2}$. We call A the dependence function of (X, Y), in accordance with the usage of Pickands (1981) and Tawn (1988a). [Do not confuse the with any other meaning of the term; for example, that of Oakes and Manatunga (1992).]

For accounts of the connections between various dependence functions, see Deheuvels (1984) and Weissman (1985).

12.5.2 Properties of Dependence Function A

1. $A(0) = A(1) = 1$.
2. $\max(w, 1-w) \le A(w) \le 1,\ 0 \le w \le 1$.
3. $A(w) = 1$ implies that X and Y are independent. $A(w) = \max(w, 1-w)$ implies that X and Y are equal, i.e., $\Pr(X = Y) = 1$.
4. A is convex; i.e., $A[\lambda x + (1-\lambda)y] \le \lambda A(x) + (1-\lambda)A(y)$.
5. If A_i are dependence functions, so is $\sum_{i=1}^n \alpha_i A_i$, where $\alpha_i \ge 0$ and $\sum_{i=1}^n = 1$.

A may or may not be differentiable. In the former case, H has a joint density everywhere; in the latter, H has a singular component and is not differentiable in a certain region of its support. We shall consider examples of this family of distributions classified as differentiable, nondifferentiable, or Tawn's extension of differentiable. Examples 1–4, 6, and 7 below were discussed by Tawn (1988a).

Nadarajah et al. (2003) studied the local dependance functions for the extreme-value distribution with dependence function A given in (12.29) and (12.30) above.

12.5.3 Differentiable Models

Example 1

The mixed model, also known as Gumbel's type A bivariate extreme-value distribution, sets $A(w) = \theta w^2 - \theta w + 1$ for $0 \le \theta \le 1$. Hence

$$\bar{H}(x, y) = \exp\left[-(x+y) + \frac{\theta xy}{x+y}\right]; \tag{12.32}$$

see Gumbel and Mustafi (1967) for further properties.

In the case of marginals with different parameters, Elandt-Johnson (1978) showed that the two crude hazard rates $h_1(x) = \frac{\partial \bar{H}}{\partial x}|_{y=x}$ and $h_2(y) = \frac{\partial H}{\partial y}|_{x=y}$ are proportional if and only if the marginal hazard rates $f/\bar{F}$ and $g/\bar{G}$ are proportional.

Example 2

The logistic model, also known as the type B extreme value-distribution, sets $A(w) = [(1-w)^r + w^r]^{1/r}$ for $r \ge 1$. Hence,

$$\bar{H}(x, y) = \exp[-(x^r + y^r)^{1/r}]. \tag{12.33}$$

This is the third type of exponential distribution mentioned, albeit only briefly, by Gumbel (1960); see Gumbel and Mustafi (1967) for further details.

12.5.4 Nondifferentiable Models

Example 3

The biextremal model, also known as the type C bivariate extreme-value distribution, sets $A(w) = \max(w, 1 - \theta w)$ for $0 \leq \theta \leq 1$. Hence,

$$\bar{H}(x, y) = \exp\left\{-\max[x + (1 - \theta)y, y]\right\}. \tag{12.34}$$

Example 4

Gumbel's model sets $A(w) = \max[1 - \theta w, 1 - \theta(1 - w)]$ for $0 \leq \theta \leq 1$. Hence,

$$\bar{H}(x, y) = [-(1 - \theta)(x + y) - \theta \max(x, y)]. \tag{12.35}$$

This is effectively the bivariate exponential distribution of Marshall and Olkin (1967) discussed in Section 10.5.

Example 5

The natural model sets $A(w) = \frac{\beta - 1}{\beta - \alpha} \max(1 - w, \alpha w) + \frac{1 - \alpha}{\beta - \alpha} \max(1 - w, \beta w)$ for $0 \leq \alpha \leq 1 \leq \beta < \infty$. Hence,

$$\bar{H}(x, y) = \exp\{-[(\beta - 1)\max(x, \alpha y) + (1 - \alpha)\max(x, \beta y)]/(\beta - \alpha)\}. \tag{12.36}$$

12.5.5 Tawn's Extension of Differentiable Models

Background

In the dependence functions for the differential models, Tawn (1988a) added an extra parameter ϕ to give further flexibility. This gives us two new models, as follows.

Example 6

The asymmetric mixed model sets $A(w) = \phi w^3 + \theta w^2 - (\theta + \phi)w + 1$ for $\theta \geq 0$, $\theta + \phi \leq 1$, $\theta + 2\phi \leq 1$, $\theta + 3\phi \geq 0$. Hence,

$$\bar{H}(x, y) = \exp\left[-(x+y) + xy\frac{(\theta+\phi)x + (\theta+2\phi)y}{(x+y)^3}\right]. \qquad (12.37)$$

When $\phi = 0$, we get the mixed model presented in Example 1.

Example 7

The asymmetric logistic model sets $A(w) = [\theta^r(1-w)^r + \phi^r w^r]^{1/r} + (\theta - \phi)w + 1 - \theta$ for $0 \leq \theta \leq 1$, $0 \leq \phi \leq 1$, $r \geq 1$. Hence,

$$\bar{H}(x, y) = \exp[-(1-\theta)x - (1-\phi)y - (\theta^r x^r + \phi^r y^r)^{1/r}]. \qquad (12.38)$$

When $\theta = \phi = 1$, we get the logistic model presented in Example 2. When $\theta = 1$, we have the biextremal model presented in Example 3, and when $\theta = \phi$ we have Gumbel's model presented in Example 4.

If $r \to \infty$, we get

$$A(w) = \max[1 - \phi w, 1 - \theta(1-w)], \qquad (12.39)$$

a nondifferentiable model with $\Pr(Y = \frac{\theta}{\phi}X) = \frac{\theta\phi}{\theta+\phi-\theta\phi}$. If $\theta = \phi = 1$, (12.39) reduces to the complete dependence model.

12.5.6 Negative Logistic Model of Joe

Joe (1990) generalized the asymmetric logistic model in Example 7 by allowing r to be negative.

Example 8

$$\bar{H}(x, y) = \exp[-(1-\theta)x - (1-\phi)y - (\theta^r x^r + \phi^r y^r)^{1/r}], \quad r < 0. \qquad (12.40)$$

X and Y are independent if $r \to 0$ and are completely dependent if $r \to \infty$ and $\theta = \phi$. $A(w)$ has the same expression as in Example 7.

12.5.7 Normal-Like Bivariate Extreme-Value Distributions

Example 10

Smith (1991) and Hüsler and Reiss (1989) considered a normal-like bivariate extreme-value distribution with exponential marginals

$$\exp\left[-x\Phi\left(\lambda + \frac{1}{2\lambda}\log\frac{x}{y}\right) - y\Phi\left(\lambda + \frac{1}{2\lambda}\log\frac{x}{y}\right)\right], \quad \lambda \geq 0, \qquad (12.41)$$

where $\Phi(x)$ is the standard normal distribution function.

12.5.8 Correlations

X and Y are positively correlated. In fact, as was pointed out by Tawn (1988a), they also have the right-tail increasing (RTI) property; see also Section 3.4.3 for this concept of positive dependence.

Pearson's product-moment correlation may be written as

$$\rho = \int_0^1 \frac{dw}{A(w)^2} - 1 \qquad (12.42)$$

[Tawn (1988a)].

For Example 1 [Tawn (1988a)], we have

$$\rho = \frac{\sin^{-1}(\frac{1}{2}\sqrt{\theta}) - \frac{1}{2}\sqrt{\theta(1-\frac{1}{4}\theta)}(1-\frac{1}{2}\theta)}{\sqrt{\theta(1-\frac{1}{4}\theta)^3}}. \qquad (12.43)$$

For Example 2 [Tawn (1988a)], we have

$$\rho = \frac{[\Gamma(1/r)]^2}{r\Gamma(2/r)} - 1. \qquad (12.44)$$

For Example 3, $\operatorname{corr}(-\log X, -\log Y)$ (i.e., the correlation when the marginals are Gumbel's extreme-value distribution) is

$$-6\pi^{-2}\int_0^\theta (1-t)^{-1}\log t\, dt, \qquad (12.45)$$

which may also be written as $6\pi^{-2}\operatorname{diln}(\theta) + 1$.[1]

[1] This is the nomenclature and notation of Spanier and Oldham (1987, p. 231). Both are different from the usage in the key book on the subject by Lewin (1981). A FORTRAN algorithm for

For Example 4, we have

$$\rho = \theta/(2-\theta). \tag{12.46}$$

See Tiago de Oliveira (1980, 1984) for the correlation coefficients when the marginals are Gumbel's extreme-value distributions in the cases of Examples 1–5; Tiago de Oliveira (1975b) gives the results for the first four, while Section 12.4 gives the first three.

As to Spearman's rho, for Example 1, it is given in (12.11). For Example 4, it is

$$\rho_S = 3\theta/(2+\theta). \tag{12.47}$$

Tiago de Oliveira (1984) gives expressions for Kendall's tau in the case of Examples 2 and 5.

Tawn (1988a) suggests $2[1-A(\frac{1}{2})]$ as another measure of dependence that is unaffected by the choice of marginals.

12.6 Bivariate Extreme-Value Distributions with Fréchet Marginals

The marginal we considered is the Fréchet distribution with $F(x) = \exp\{-x^{-1}\}$, $x > 0$. A simple transformation $Z = X^{-1}$ yields a unit exponential distribution.

Kotz and Nadarajah (2000) considered a bivariate extreme value distribution with the distribution function written as in (12.29) with Fréchet marginals instead of the exponentials as given by

$$H(x,y) = \exp\left[-\left(\frac{1}{x}+\frac{1}{y}\right)A\left(\frac{x}{x+y}\right)\right], \quad x,y>0, \tag{12.48}$$

where $A(w) = \int_0^1 \max[(1-w)q, w(1-q)]\frac{dB}{dq}dq$, as expressed in (12.30). Instead of using the dependence function A, the bivariate extreme-value distribution is now characterized by $\frac{dB}{dq} = b(q)$.

12.6.1 Bilogistic Distribution

Example 10

Joe et al. (1992) considered

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calculating the dilogarithm in Lewin's sense has been published by Ginsberg and Zaborowski (1975).

$$H(x,y) = \exp\left[-\int_0^1 \max\left\{\frac{(q_1-1)s^{-1/q_1}}{q_1 x}, \frac{(q_2-1)s^{-1/q_2}}{q_2 y}\right\} ds\right] \quad (12.49)$$

for $q_1 > 0$ and $q_2 > 0$. Here, we have

$$b(w) = \frac{(1-1/q_1)(1-z)z^{1-1/q_1}}{(1-w)w^2\{(1-z)/q_1 + z/q_2\}},$$

where z is the root of the equation

$$(1-1/q_1)(1-w)(1-z)^{1/q_2} - (1-1/q_2)wz^{1/q_1} = 0.$$

12.6.2 Negative Bilogistic Distributions

Example 11

Coles and Tawn (1994) considered a family of distributions having the same distribution function as in Example 8 except that $q_1 < 0$ and $q_2 < 0$ and

$$b(w) = -\frac{(1-1/q_1)(1-z)z^{1-1/q_1}}{(1-w)w^2\{(1-z)/q_1 + z/q_2\}}, \quad q_1 < 0, q_2 < 0.$$

12.6.3 Beta-Like Extreme-Value Distribution

Example 12

Coles and Tawn (1991) considered a beta-like bivariate extreme-value distribution with cumulative distribution function

$$H(x,y) = \exp\left[-\frac{1}{x}\{1 - B_u(q_1+1, q_2)\} - \frac{1}{y}B_v(q_1, q_2+1)\right], \quad q_1 > 0,\ q_2 > 0, \quad (12.50)$$

where $u = \frac{q_1 x}{q_1 x + q_2 y}$, $v = \frac{q_1 y}{q_1 x + q_2 y}$, and

$$B_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x w^{a-1}(1-w)^{b-1} dw.$$

In this case,

$$b(w) = \frac{q_1^{q_1} q_2^{q_2} \Gamma(q_1+q_2+1)}{\Gamma(q_1)\Gamma(q_2)} \frac{w^{q_1-1}(1-w)^{q_2-1}}{\{q_1 w + q_2(1-w)\}^{1+q_1+q_2}}, \quad w \in (0,1).$$

12.7 Bivariate Extreme-Value Distributions with Weibull Marginals

This distribution was studied by Oakes and Manatunga (1992).

12.7.1 Formula of the Cumulative Distribution Function

The joint distribution function is

$$\bar{H}(x,y) = \exp[-\{(\eta_1^{\kappa_1} x^{\kappa_1})^{\phi} + (\eta_2^{\kappa_2} y^{\kappa_2})^{\phi}\}^{\alpha}. \tag{12.51}$$

Here, the parameter $\alpha = 1/\phi$ represents the degree of dependence between X and Y, and $\alpha = 1 - \tau$ is Kendall's coefficient of concordance. Cases $\alpha = 0$ and $\alpha = 1$ correspond to maximal positive dependence and independence, respectively.

12.7.2 Univariate Properties

The marginal survival functions are

$$\bar{F}(x) = \exp(-\eta_1^{\kappa_1} x^{\kappa_1}), \quad \bar{G}(y) = \exp(-\eta_2^{\kappa_2} y^{\kappa_2}), \quad x, y \geq 0,$$

with scale parameters η_1 and η_2 and shape parameters κ_1 and κ_2, respectively.

12.7.3 Formula of the Joint Density

The joint density function is

$$h(x,y) = \phi\kappa_1\kappa_2\eta_1^{\kappa_1\phi}\eta_2^{\kappa_2\phi}x^{\kappa_1\phi-1}y^{\kappa_2\phi-1}s^{\alpha-2}(1-\alpha+\alpha z)e^{-z}, \tag{12.52}$$

where

$$s = (\eta_1^{\kappa_1} x^{\kappa_1})^{\phi} + (\eta_2^{\kappa_2} y^{\kappa_2})^{\phi}, \quad z = s^{\alpha}.$$

12.7.4 Fisher Information Matrix

Using Lee's (1979) transformation, Oakes and Manatunga (1992) have derived an explicit formula for the elements of the Fisher information matrix for this distribution.

12.7.5 Remarks

- Oakes and Manatunga (1992) numerically calculated the asymptotic variance of the maximum likelihood estimator $\hat{\alpha}$ of α. Calculations reveal that estimators of the scale parameters η_1 and η_2 are almost orthogonal to that of the dependence parameter α.
- By marginal transformation to Gumbel marginals and reparametrizing such that $\tau_1 = \kappa_1^{-1}$ and $\tau_2 = \kappa_2^{-1}$, Shi et al. (2003) have shown that the bivariate Weibull model in (12.51) reduces to type B (the logistic model) with scale parameters τ_1 and τ_2. Thus, testing for $\kappa_1 = \kappa_2$ of the bivariate Weibull becomes testing for the equality of the scale parameters τ_1 and τ_2 of the type B distribution.

12.8 Methods of Derivation

- Bivariate extreme-value distributions arise as the limiting distributions of normalized componentwise maxima. More formally, let (X_i, Y_i), $i = 1, 2, \ldots, n$, be i.i.d. random vectors. Then, $(\max(X_i), \max(Y_i))$, after being suitably normalized, has a bivariate extreme-value distribution.
- (X, Y) has a bivariate extreme-value distribution with unit exponential marginals if and only if the marginals are unit exponentials and, for every $n \geq 1$, $[\bar{H}(x, y)]^n = \bar{H}(nx, ny)$. Pickands (1981) showed that this equation is satisfied if and only if $\bar{H}(x, y)$ can be written as (12.29). For this reason, the dependence function determines the type of bivariate extreme-value distribution; it also expresses the asymptotic connection between two maxima.
- Alternatively, (X, Y) has a bivariate extreme-value distribution with unit exponential marginals if and only if $\min(aX, bY)$ is exponential for all $a, b > 0$.

12.9 Estimation of Parameters

Kotz et al. (2000, Chapter 53) discusses estimation of the parameters of type A, B, and C distributions. Kotz and Nadarajah (2000) have devoted their Section 3.6 to estimation problems for multivariate extreme distributions. Shi (1995a) discussed moment estimation for the logistic model whereas Shi and Feng (1997) considered the maximum likelihood and stepwise method for the parameters of the logistic model.

12.10 References to Illustrations

Plots of the bivariate density along $y = x$ of the mixed and logistic models in Examples 1 and 2, with their marginals being of extreme value of type I form, are given by Gumbel and Mustafi (1967) and Kotz et al. (2000, p. 631). Density and density contour plots of type A and type B (with Gumbel marginals) are given by Arnold et al. (1999, pp. 283–284).

12.11 Generation of Random Variates

Section 3.7 of Kotz and Nadarajah (2000) has given three known methodologies for simulating bivariate extreme-value observations.

12.11.1 Shi et al.'s (1993) Method

Shi et al. (1993) described a scheme for simulating (X, Y) from the bivariate symmetric logistic distribution (type B) as given in (12.12); i.e., $H(x, y) = \exp\left[-(e^{-qx} + e^{-qy})^{1/q}\right]$. Letting $X = Z\cos^{2/q} V$ and $V = Z\sin^{2/q} V$, they observed that the joint density of (U, V) can be factorized as

$$(q^{-1}z + 1 - q^{-1})e^{-z}\sin 2v, \quad 0 < v < \pi/2,\ 0 < z < \infty,$$

which shows that Z and V are independent. It is then shown that V may be represented as $\arcsin U^{1/2}$, where U is uniform on $(0, 1)$, whereas Z is a mixture of two independent exponentials with a ratio $1 - q^{-1} : q^{-1}$. We can now see that (12.12) can be simulated easily.

12.11.2 Ghoudi et al.'s (1998) Method

Ghoudi et al. (1998) described a simulation scheme that is applicable for all bivariate extreme-value distributions. Starting with the expression for the cumulative distribution of the copula associated with the bivariate extreme-value distribution given by (12.1), Ghoudi et al. (1998) first find the joint distribution of $Z = X/(X+Y)$ and $V = A(-X, -Y)$ and then the marginal distribution of Z and the conditional distribution of V given $Z = z$. From these, one can simulate (X, Y), of course!

12.11.3 Nadarajah's (1999) Method

Nadarajah (1999) used the limiting point process result as an approximation to simulate bivariate extreme values.

12.12 Applications

Extreme-value distributions have wide applications in environmental studies (earthquake magnitudes, floods, river flows, storm rainfalls, wind speeds, etc.), insurance and finance, structural design, and telecommunications. There are several books that are devoted to applications of extreme-value distributions; see, for example, Tawn (1994), Embrechts et al. (1997), Kotz and Nadarajah (2000), and Coles (2001). For a more recent survey article, one may refer to Smith (2003).

12.12.1 Applications to Natural Environments

- In the form with extreme-value marginals, the mixed model in Example 1 and the logistic model in Example 2 were both used by Gumbel and Mustafi (1967) to describe the flood of the Fox River at upstream and a downstream gauging stationd. They found the latter fitted better; see Gumbel and Goldstein (1964) for floods of the Ocmulgee River. Tiago de Oliveira (1975b, 1980) mentions that an unpublished paper of Amaral and Gomes in 1975 entitled "The fitting of bivariate extreme models" has reanalyzed these and other datasets.
- The models of Examples 1, 2, 6, and 7 were used by Tawn (1988a) to describe the annual maximum sea levels at Lowestoft and Sheerness.
- Smith (1986) and Tawn (1988b) considered the joint distribution of the r largest observations—they had time series data of sea level, and were

concerned with issues such as the improvement in prediction resulting from making use of the five or ten largest values per year rather than only the largest. Smith's data were from Venice, and Tawn's were from Lowestoft and Great Yarmouth.

- The "station-year" method for the analysis of rainfall or flood maxima is motivated as follows. One may be interested in events with very long return periods (i.e., well out in the tail of the distribution), much larger events than the lengths of the individual rainfall datasets. To make deductions about such rare events, one might wish to combine all datasets from measuring stations in a region to form a single series. The extent to which this is justified depends on the tail of the joint distribution of the rainfall amounts; see Buishand (1984), who considered the ratio $q = \log H(x, x)/\log F(x)$. In the case of independence, this ratio is 2. For annual maximum daily rainfall data from the Netherlands, Buishand plotted q against F for pairs of stations different distances apart. The ratio q increases with both F and distance and seems to be tending to 2. For data restricted to the winter season, the results were more complex.
- Lewis (1975) has briefly mentioned work by himself and Daldry on annual maxima of wind and gust.
- Smith (1991) applied the normal-like bivariate extreme-value distribution to model spatial variations of extreme storms at two locations.
- Coles and Tawn (1994) found the negative bilogistic distribution most suitable for estimating the dependence between the extremes of surge and wave height.
- Yue (2000) used the type A model with Gumbel marginals to model a multivariate storm event, 104-yr daily rainfall data at the Niigata observation station in Japan during 1897–1990.
- Yue (2001) used a type 1 bivariate extreme-value distribution (the logistic model) with Gumbel marginals as a joint distribution of annual maximum storm peaks (maximum rainfall intensities) and the corresponding storm amounts. The model was found to fit well to the rainfall data collected from the Tokushima Meteorological Station of Tokushima Prefecture, Japan.
- In analyzing flood frequency of a region in Northwestern Mexico, Escalante-Sandoval (2007) used a (i) type B bivariate extreme-value two-parameter Weibull distributionas marginals and (ii) type B distribution with mixtures of two Weibull distributions as its marginals. See also Escalante-Sandoval (1998).

A salutary quotation [Klemeš (1987)] is as follows: "The natural frequencies of flood peaks in the historic series are, in fact, almost never analyzed. We do not learn whether there seems to be any pattern such as clustering of high or low peaks, trend, or some other feature, nor any indication of some hydrological, geographical or other context that could shed light on the historic flood record. What happens is that the actual time sequencing is completely ignored and the flood record is declared purely random. The ostensible reason for this is to 'simplify the mathematical treatment.' This, however, is rather amusing when one sees how the laudable resolve to keep things simple is then hastily abandoned and the use of the most advanced theories is advocated for the treatment of this artificially random sample on the pretext that 'greatest amount of information' must be extracted."

12.12.2 Financial Applications

One of the driving forces for the popularity of copulas, especially the extreme-value copulas, is their application in the context of financial risk management. Mikosch (2006, Section 3) explains the reasons why the finance researchers are attracted to copulas. Section 1.15.1 provides a list of applications in this area.

12.12.3 Other Applications

There are many other applications of extreme-value copulas as given in Section 1.15. In addition to what have already been described in Chapter 1, we give the following examples.

- In the form with extreme-value marginals, the mixed model in Example 1 was used by Posner et al. (1969) in their analysis of a spacecraft command receiver.
- The logistic model was also used by Hougaard (1986) to analyze data on tumors in rats.
- For applications to structural design, see Coles and Tawn (1994). Kotz and Nadarajah (2000, p. 145) reanalyzed the Swedish data of ages at death classified according to gender up to year 1997. The result confirms the original finding of independence studied by Gumbel and Goldstein (1964).

12.13 Conditionally Specified Gumbel Distributions

Introducing location and scale parameters, the univariate Gumbel extreme-value distribution (for maxima) has a density of the form

$$f(x) = \frac{1}{\sigma}e^{-(x-\mu)/\sigma}\exp\left(-e^{-(x-\mu)/\sigma}\right), \quad -\infty < x < \infty, \tag{12.53}$$

where μ and σ are, respectively, the location and scale parameters. Chapter 12 of Arnold et al. (1999) considered conditional distributions rather than marginals that are of the Gumbel form.

12.13.1 Bivariate Model Without Having Gumbel Marginals

Section 12.3 of Arnold et al. (1999) considered two conditionally specified Gumbel distributions, neither of them valid bivariate extreme-value distributions. After repametrizations and standardization, we have the following.

Formula of the Joint Density

The joint density function is

$$h(x,y) = k(\theta)\exp[-x - y - e^{-x} - e^{-y} - \theta e^{-x-y}], \tag{12.54}$$

where the normalizing constant is given by

$$k(\theta) = \frac{-}{\theta e^{-1/\theta}} - \mathrm{Ei}(1/\theta), \tag{12.55}$$

in which θ is a dependency parameter and $-\mathrm{Ei}(t) = \int_t^\infty \frac{e^{-u}}{u}\,du$.

Univariate Properties

The marginal density of X is

$$f(x) = k(\theta)\frac{\exp[-x - e^{-x}]}{1 + \theta e^{-x}}, \tag{12.56}$$

and a similar expression holds for $g(y)$.

Conditional Properties

The conditional density of X, given $Y = y$, is

$$f(x|y) = (1 + \theta e^{-y})\exp[-x - e^{-x}(1 + \theta e^{-y})] \tag{12.57}$$

and

$$g(y|x) = (1 + \theta e^{-x})\exp[-y - e^{-y}(1 + \theta e^{-x})]. \tag{12.58}$$

Correlations and Dependence

Arnold et al. (1999) have shown that (12.54) is always totally negative of order 2 (also known as RR_2 in Section 3.8), and consequently the correlations are negative.

References to Illustrations

Arnold et al. (1999, p. 285) have presented a density plot and a density contour plot.

12.13.2 Nonbivariate Extreme-Value Distributions with Gumbel Marginals

Arnold et al. (1999) derived another nonvalid bivariate extreme-value distribution by conditional specification as given below (in its standardized form). The specification is through conditional distribution functions rather than conditional densities.

Formula for Cumulative Distribution Function

The joint distribution function is

$$H(x, y) = \exp[-e^{-x} - e^{-y} - \theta e^{-x-y}], \quad 0 < \theta < 1. \tag{12.59}$$

Formula for the Joint Density

The joint density function is

$$\begin{aligned} h(x, y) = \exp(&-e^{-x} - e^{-y} - \theta e^{-x-y} - x - y) \\ &\times [(1 + \theta e^{-x})(1 + \theta e^{-y}) - \theta]. \end{aligned} \tag{12.60}$$

Univariate Properties

Both marginal distributions are Gumbel distributions.

Conditional Properties

We have

$$\Pr(X \leq x | Y \leq y) = \exp[-e^{-x}(1 + \theta e^{-y})], \tag{12.61}$$

which is also Gumbel.

Correlations and Dependence

Arnold et al. (1999) have shown that X and Y are NQD and hence have a negative correlation.

References to Illustrations

A density plot and density contour plot of (12.60) are given in Arnold et al. (1999, p. 285).

12.13.3 Positive or Negative Correlation

Tiago de Oliveira (1962) showed that every bivariate extreme model exhibits a non-negative correlation. This result also follows from the fact that X and Y are PQD (see Section 12.2.2), and so they must be positively correlated.

Arnold et al. (1999, p. 282) made the following remark:

> "However, many bivariate data sets are not associated with maxima of sequences of i.i.d. random vectors even though marginally and /or conditionally a Gumbel model may fit quite well.
>
> "Quite often empirical extreme data are associated with dependent bivariate sequences. Unless the dependence is relatively weak, there is no reason to expect the classical bivariate extreme theory will apply in such settings and consequently no a priori argument in favor of non-negative or nonpositive correlation.
>
> "The conditionally specified Gumbel models introduced in this chapter exhibit nonpositive correlations. Thus, the Gumbel–Mustafi models and the conditionally specified models do not compete but, in fact, complement each other. Together they provide us with the ability to fit data sets exhibiting a broad spectrum of correlation structure, both negative and positive."

12.13.4 Fields of Applications

Simiu and Filliben (1975) presented data on annual maximal wind speeds at 21 locations in the United States. About 40% of the 210 pairs of stations in this dataset exhibit negative correlation, so the phenomenon is not an isolated one. Thus, a bivariate extreme-value distribution is not appropriate. Arnold et al. (1999) found that (12.55) and (12.61) provide a good fit to data from two stations, Eastport and North Head.

References

1. Arnold, B.C., Castillo, E., Sarabia, J.M.: Conditional Specification of Statistical Models. Springer-Verlag, New York (1999)
2. Beirlant, J., Goegebeur, Y., Segers, J., Teugels, J.: Statistics of Extreme: Theory and Applications. John Wiley and Sons, Chichester (2004)
3. Buishand, T.A.: Bivariate extreme-value data and the station-year method. Journal of Hydrology **69**, 77–95 (1984)
4. Coles, S.G.: An Introduction to Statistical Modeling of Extreme Values. Springer-Verlag, New York (2001)
5. Coles, S.G., Tawn, J.A.: Modelling extreme multivariate events. Journal of the Royal Statistical Society, Series B **53**, 377–392 (1991)
6. Coles, S.G., Tawn, J.A.: Statistical methods for multivariate extremes: An application to structural design. Applied Statistics **43**, 1–48 (1994)
7. de Haan, L., Resnick, S.I.: Limit theory for multivariate sample extremes. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **40**, 317–337 (1977)
8. Deheuvels, P.: Point processes and multivariate sample extreme values. Journal of Multivariate Analysis **13**, 257–271 (1983)
9. Deheuvels, P.: Probabilistic aspects of multivariate extremes. In: Statistical Extremes and Applications, J. Tiago de Oliveira (ed.) pp.117–130. Reidel, Dordrecht (1984)
10. Deheuvels, P.: Point processes and multivariate extreme values (II). In: Multivariate Analysis-VI, P.R. Krishnaiah (ed.) pp. 145–164. North-Holland, Amsterdam (1985)
11. Elandt-Johnson, R.C.: Some properties of bivariate Gumbel Type A distributions with proportional hazard rates. Journal of Multivariate Analysis **8**, 244–254 (1978)
12. Embrechts, P., Klüppelberg, C., Mikosch, T.: Modeling Extremal Events for Insurance and Finance. Springer-Verlag, New York (1997)
13. Escalante-Sandoval, C.: Multivariate extreme value distribution with mixed Gumbel marginals. Journal of the American Water Resources Association **34**, 321–333 (1998)
14. Escalante-Sandoval, C.: Application of bivariate extreme value distribition to flood frequency analysis: A case study of Northwestern Mexico. Natural Hazards **47**, 37–46 (2007)
15. Galambos, J.: The Asymptotic Theory of Extreme Order Statistics, 2nd edition. Kreiger, Malabar, Florida (1987)
16. Geffroy, J.: Contributions à la théorie des valeurs extrême. Publications de l'Institut de Statistique de l'Université de Paris **7**, 37–121, and **8**, 123–184 (1958/59)
17. Ghoudi, K., Khouddraji, A., Rivest, L.P.: Statistical properties of couples of bivariate extreme-value copulas. Canadian Journal of Statistics **26**, 187–197 (1998)
18. Ginsberg, E.S., Zaborowski, D.: Algorithm 490: The dilogarithm function of a real argument. Communications of the Association for Computing Machinery **18**, 200–202 (Remark, ACM Transactions on Mathematical Software **2**, 112) (1975)
19. Gumbel, E.J.: Distributions à plusieurs variables dont les marges sont données. Comptes Rendus de l'Académie des Sciences, Paris **246**, 2717–2719 (1958)
20. Gumbel, E.J.: Bivariate exponential distributions. Journal of the American Statistical Association **55**, 698–707 (1960)
21. Gumbel, E.J.: Two systems of bivariate extremal distributions. Bulletin of the International Statistical Institute **41**, 749–763 (Discussion, 763) (1965)

22. Gumbel, E.J., Goldstein, N.: Analysis of empirical bivariate extremal distributions. Journal of the American Statistical Association **59**, 794–816 (1964)
23. Gumbel, E.J., Mustafi, C.K.: Some analytical properties of bivariate extremal distributions. Journal of the American Statistical Association **62**, 569–588 (1967)
24. Hougaard, P.: A class of multivariate failure time distributions. Biometrika **73**, 671–678 (Correction **75**, 395) (1986)
25. Hüsler, J., Reiss, R.D.: Maxima of normal random vectors: Between independence and complete dependence. Statistics and Probability Letters **7**, 283–286 (1989)
26. Joe, H.: Families of min-stable multivariate exponential and multivariate extreme value distributions. Statistics and Probability Letters **9**, 75–81 (1990)
27. Joe, H., Smith, R.L., Weissmann, I.: Bivariate threshold method for extremes. Journal of the Royal Statistical Society, Series B **54**, 171–183 (1992)
28. Klemeš, V.: Empirical and causal models in hydrologic reliability analysis. In: Engineering Reliability and Risk in Water Resources, L. Duckstein and E.J. Plate (eds.), pp. 391–403. Nijhoff, Dordrecht (1987)
29. Kotz S., Balakrishnan, N., Johnson N.L.: Continuous Multivariate Distributions, Vol 1: Models and Applications, 2nd edition. John Wiley and Sons, New York (2000)
30. Kotz, K., Nadarajah, D.: Extreme Value Distributions: Theory and Applications. Imperial College Press, London (2000)
31. Lee, L-F.: On comparisons of normal and logistic models in the bivariate dichotomous analysis. Economics Letters **4**, 151–155 (1979)
32. Lewin, L.: Polylogarithms and Associated Functions. North-Holland, New York (1981)
33. Lewis, T.: Contribution to Discussion of paper by Tiago de Oliveira. Bulletin of the International Statistical Institute **46**, 253 (1975)
34. Marshall, A.W., Olkin, I.: A multivariate exponential distribution. Journal of the American Statistical Association **62**, 30–44 (1967)
35. Mikosch, T.: Copulas: Tales and facts. Extremes **9**, 3–20 (2006)
36. Nadarajah, S.: Simulation of multivariate extreme values. Journal of Statistical Computation and Simulation **62**, 395–410 (1999)
37. Nadarajah, S., Mitov, K., Kotz, S.: Local dependence function for extreme value distributions. Journal of Applied Statistics **30**(10), 1081–1100 (2003)
38. Nelsen, R.B.: An Introduction to Copulas, 2nd edition. Springer-Verlag, New York (2006)
39. Oakes, D., Manatunga, A.K.: Fisher Information for a bivariate extreme value distribution. Biometrika **79**, 827–832 (1992)
40. Pickands, J.: Multivariate extreme value distributions. Bulletin of the International Statistical Institute **49**, 859–878 (Discussion, 894–902) (1981)
41. Posner, E.C., Rodemich, E.R., Ashlock, J.C., Lurie, S.: Application of an estimator of high efficiency in bivariate extreme value theory. Journal of the American Statistical Association **64**, 1403–1414 (1969)
42. Resnick, S.: Extreme Values, Regular Variations, and Point Processes. Springer-Verlag, New York (1987)
43. Shi, D-J.: Moment estimation for multivariate extreme value distribution, Applied Mathematics–JCU, **10B**, 61–68 (1995a)
44. Shi D-J.: Multivariate extreme value distribution and its Fisher information matrix. Acta Mathematica Applicatae Sinica **11**, 421–428 (1995b)
45. Shi, D-J.: A property for bivariate extreme value distribution. Chinese Journal of Applied Probability and Statistics **19**, 49–54 (2003)
46. Shi, D-J., Feng, Y-J.: Parametric estimations by maximum likelihood and stepwise method for multivariate extreme value distribution. Journal of System Science and Mathematical Sciences **17**, 243–251 (1997)
47. Shi, D-J., Smith, R.K., Coles, S.G.: Joint versus marginal estimation for bivariate extremes. Technical Report No. 2074, Department of Statistics, University of North Carolina, Chapel Hill (1993)

48. Shi, D-J., Tang, A-L., Wang, L.: Test of shape parameter of bivariate Weibull distribution. Journal of Tianjin University **36**, 68–71 (2003)
49. Simiu, E., Filliben, B.: Structure analysis of extreme winds. Technical Report 868, National Bureau of Standards, Washington, D.C. (1975)
50. Smith, R.L.: Extreme value theory based on the r largest annual events. Journal of Hydrology **86**, 27–43 (1986)
51. Smith, R.L.: Extreme value theory. In: Handbook of Applicable Mathematics, Volume 7, pp. 437–472. John Wiley and Son, New York (1990)
52. Smith, R.L.: Regional estimation from spatially dependent data, Technical Report, Department of Statistics, University of North Carolina, Chapel Hill (1991)
53. Smith, R.L.: Extreme Values. Chapman and Hall, London (1994)
54. Smith, R.L.: Statistics of extremes, with applications in environment, insurance and finance. In: Extreme Values in Finance, Telecommunications, and the Environment, B. Finkenstadt and H. Rootzen (eds.), pp. 1–78. Chapman and Hall/CRC Press, London (2003)
55. Spanier, J., Oldham, K.B.: An Atlas of Functions. Hemisphere, Washington, D.C. and Springer-Verlag, Berlin (1987)
56. Tawn, J.A.: Bivariate extreme value theory: Models and estimation. Biometrika **75**, 397–415 (1988a)
57. Tawn, J.A.: An extreme-value theory for dependent observations. Journal of Hydrology **101**, 227–250 (1988b)
58. Tawn, J.: Applications of multivariate extremes. In: Extreme Value Theory and Applications. Proceedings of the Conference on Extreme Value Theory and Applications, Volume 1, J. Galambos, J. Lechner, and E. Smith (eds.), pp. 249–268. Kluwer Academic Publishers, Boston (1994)
59. Tiago de Oliveira, J.: La représentation des distributions extrêmales bivariées. Bulletin de l'Institut International de Statistique **39**, 477–480 (1961)
60. Tiago de Oliveira, J.: Structure theory of bivariate extremes, extensions. Estudos de Matematica, Estatistica, e Economicos **7**, 165–195 (1962/63)
61. Tiago de Oliveira, J.: Bivariate and multivariate extreme distributions. In: A Modern Course on Distributions in Scientific Work, Volume 1: Models and Structures, G.P. Patil, S. Kotz, and J.K. Ord (eds.), pp. 355–361. Reidel, Dordrecht (1975a)
62. Tiago de Oliveira, J.: Bivariate extremes: Extensions. Bulletin of the International Statistical Institute **46**, 241–252 (Discussion, 253–254) (1975b)
63. Tiago de Oliveira, J.: Bivariate extremes: Foundations and statistics. In: Multivariate Analysis-V, P.R. Krishnaiah (ed.) pp.349–366, North-Holland, Amsterdam (1980)
64. Tiago de Oliveira, J.: Bivariate models for extremes; Statistical decision. In: Statistical Extremes and Applications, J. Tiago de Oliveira (ed.), pp. 131–153. Reidel, Dordrecht (1984)
65. Yue, S.: The Gumbel mixed model applied to storm frequency analysis. Water Resources Management **14**, 377–389 (2000)
66. Yue, S.: The Gumbel logistic model for representing a multivariate storm event. Advances in Water Resources **24**, 179–185 (2001)
67. Yue, S., Ouarda, T.B.M.J., Bobée, B., Legendre, P., Bruneau, P.: The Gumbel mixed model for flood frequency analysis. Journal of Hydrology **226**, 88–100 (1999); see Corrigendum **228**, 283 (2000)
68. Yue, S., Wang, C.Y.: A comparison of two bivariate extreme value distributions. stochastic Environmental Research **18**, 61–66 (2004)
69. Weissman, I.: Multidimensional extreme value theory (Discussion). Bulletin of the International Statistical Institute **51**, 187–190 (1985)

Chapter 13
Elliptically Symmetric Bivariate Distributions and Other Symmetric Distributions

13.1 Introduction

This chapter is devoted to describing a class of bivariate distributions whose contours of probability densities are ellipses; in particular, those ellipses with constant eccentricity. These distributions are generally known as elliptically contoured or elliptically symmetric distributions. A subclass of distributions with contours that are circles are known as spherically symmetric (or simply spherical) distributions. The chapter also includes other symmetric bivariate distributions.

The last 20 years have seen vigorous development of multivariate elliptical distributions as direct generalizations of the multivariate normal distribution that dominated statistical theory and applications for nearly a century. Elliptically contoured distributions retain most of the attractive properties of the multivariate normal distribution. For example, let (X, Y) be an uncorrelated pair from this class. Then, X^2/Y^2 has the usual F-distribution, and $X^2/(X^2 + Y^2)$ has the beta distribution, beta$(\frac{1}{2}, \frac{1}{2})$; see Kelker (1970).

On the application side, members of this class were used to describe the second-order moments of the transformation of a random signal by an instantaneous linear device [McGraw and Wagner (1968)]. Further, van Praag and Wesselman (1989) have shown that many procedures for multivariate analysis in the normal case can be adapted to the elliptical case with the aid of the estimated kurtosis. Bentler and Berkane (1985) went as far as to say, "It is becoming apparent that [elliptical] theory has a potential to displace multivariate normal theory in a variety of applications such as linear structural modelling" (which includes factor analysis and simultaneous equation models). For an early review and bibliography of these distributions, see Chmielewski (1981).

Fang et al. (1990) provided a rather detailed study of these distributions, and their text has now become a standard reference for symmetric multivariate distributions. A more recent review is Fang (1997).

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