Fundamental of Extreme Value Theory

We review some fundamentals of extreme value theory which concern stochastic behavior of the extreme values in a single process. We illustrate the power of the theory by means of four applications to climate data from different parts of the world: rainfall data from Florida, wind data from New Zealand, rainfall data from South Korea and flood data from Taiwan.

1. INTRODUCTION

Extreme value theory deals with the stochastic behavior of the extreme values in a process. For a single process, the behavior of the maxima can be described by the three extreme value distributions–Gumbel, Fréchet and negative Weibull–as suggested by Fisher and Tippett (1928). Kotz and Nadarajah (2000) indicated that the extreme value distributions could be traced back to the work done by Bernoulli in 1709. The first application of extreme value distributions was probably made by Fuller in 1914. Thereafter, several researchers have provided useful applications of extreme value distributions particularly to climate data from different regions of the world. For a review of applications see Farago and Katz (1990).

2. EXTREME VALUE THEORY

Suppose $X_1, X_2, \ldots$ are independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) F. Let $M_n = \max\{X_1, \ldots, X_n\}$ denote the maximum of the first n random variables and let $w(F) = \sup\{x : F(x) < 1\}$ denote the upper end point of F. Since

$$\Pr(M_n \leq x) = \Pr(X_1 \leq x, \ldots, X_n \leq x) = F^n(x),$$

M_n converges almost surely to $w(F)$ whether it is finite or infinite. The limit theory in univariate extremes seeks norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \to G(x) \tag{1}$$

as $n \to \infty$. If this holds for suitable choices of a_n and b_n then we say that G is an extreme value cdf and F is in the domain of attraction of G, written as $F \in D(G)$. We say further that two extreme value cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x. The Extremal Types Theorem (Fisher and Tippett, 1928; Gnedenko, 1943; de Haan, 1970, 1976; Weissman, 1978) characterizes the limit cdf G as of the type of one of the following three classes:

$$I \;:\; \Lambda(x) = \exp\{-\exp(-x)\}, \qquad x \in \Re;$$

$$II \;:\; \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \geq 0 \end{cases}$$

$$\text{for some } \alpha > 0; \tag{2}$$

$$III \quad : \quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\} & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

$$\text{for some } \alpha > 0.$$

Thus, any extreme value distribution can be classified as one of Type I, II or III. The three types are often called the Gumbel, Fréchet and Weibull types, respectively.

Leadbetter *et al.* (1983) gave a comprehensive account of necessary and sufficient conditions for $F \in D(G)$ and characterizations of a_n and b_n when G is one of the three extreme value cdfs above. The necessary and sufficient conditions for the three cdfs in (2) are:

$$\begin{aligned} I \quad &: \quad \exists \gamma(t) > 0 \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\ II \quad &: \quad w(F) = \infty \text{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\ III \quad &: \quad w(F) < \infty \text{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{\alpha}, \qquad x > 0. \end{aligned}$$

The corresponding characterizations of a_n and b_n are:

$$\begin{aligned} I \quad &: \quad a_n = \gamma\left(F^{\leftarrow}\left(1 - n^{-1}\right)\right) \text{ and } b_n = F^{\leftarrow}\left(1 - n^{-1}\right), \\ II \quad &: \quad a_n = F^{\leftarrow}\left(1 - n^{-1}\right) \text{ and } b_n = 0, \\ III \quad &: \quad a_n = w(F) - F^{\leftarrow}\left(1 - n^{-1}\right) \text{ and } b_n = w(F), \end{aligned}$$

where $F^{\leftarrow}$ denotes the inverse function of F. Note that only distributions with $w(F) = \infty$ (respectively, $w(F) < \infty$) can qualify for membership in $D(\Phi_\alpha)$ (respectively, $D(\Psi_\alpha)$).

An equivalent characterization of G is by means of the definition of max-stability: a df G is max-stable if there exists $\alpha_n > 0, \beta_n$ such that for each $n \geq 1$

$$G^n(x) = G(\alpha_n x + \beta_n). \tag{3}$$

It can be shown (Resnick, 1987, Proposition 5.9) that the class of extreme value dfs G is precisely the class of nondegenerate max-stable dfs. It is easily checked that α_n and β_n for the three dfs in (2) are:

$$\begin{aligned} I \quad &: \quad \alpha_n = 1 \text{ and } \beta_n = -\log n, \\ II \quad &: \quad \alpha_n = n^{-1/\alpha} \text{ and } \beta_n = 0, \\ III \quad &: \quad \alpha_n = n^{1/\alpha} \text{ and } \beta_n = 0. \end{aligned} \tag{4}$$

Various methods have been developed to test whether a sequence of iid observations belong to the domain of attraction of one the three distributions (see Tiago de Oliveira and Gomes (1984) and Marohn (1998a, 1998b) for the Gumbel type; Tiku and Singh (1981) and Shapiro and Brain (1987a, 1987b) for the Weibull type; Galambos (1982), Öztürk (1986), Öztürk and Korukoğlu (1988), Castillo *et al.* (1989) and Hasofer and Wang (1992) for general tests).

Anderson (1970, 1980) and Anderson *et al.* (1997) develop limit laws corresponding to (2) for discrete random variables. For instance, Anderson (1980) shows that if X_i are discrete random variables then under certain conditions either there exist $a_n > 0$ and b_n such that

$$\lim_{n\to\infty} a_n \Pr\left(M_n = a_n\left(x + o(1)\right) + b_n\right) = \exp\left\{-\exp(-x)\right\}$$

or there exist $a_n > 0$ such that

$$\lim_{n\to\infty} a_n \Pr\left(M_n = a_n\left(x + o(1)\right)\right) = \exp\left(-x^{-\alpha}\right).$$

Other developments on the extremes of discrete random variables include Arnold and Villaseñor (1984) and Gordon *et al.* (1986) who consider sequences $X_1, \ldots, X_n$ of independent Bernoulli random variables and study the limiting behavior for a fixed $l \geq 0$ of the longest l-interrupted head run, i.e., the maximal value of $k - i$, where $0 \leq i < k \leq n$, and where there are exactly l values of $i < m < k$ such that $X_m = 0$ (tails).

Pancheva (1984) extends (2) for power normalization to obtain what is known as the p-max stable laws. Namely, she seeks constants $a_n > 0$ and $b_n > 0$ such that

$$\lim_{n\to\infty} \Pr\left\{\left|\frac{M_n}{a_n}\right|^{1/b_n} \operatorname{sign}\left(M_n\right) \leq x\right\} = \lim_{n\to\infty} F^n\left(a_n \mid x \mid^{b_n} \operatorname{sign}(x)\right) = G(x)$$

for all $x \in \mathcal{C}(G)$, the set of continuity points of G, where $\operatorname{sign}(x) = -1, 0, 1$ according as $x < 0$, $x = 0$ or $x > 0$. Then it is shown that G must be of the same p-type (G and G^* are of the same p-type if there exists constants $a > 0$ and $b > 0$ such that $G(x) = G^*\left(a \mid x \mid^b \operatorname{sign}(x)\right)$ for all x real) as one of the following six dfs

$$\begin{aligned}
G(x) &= \begin{cases} 0 & \text{if } x \leq 1, \\ \exp\left\{-(\log x)^{-\alpha}\right\} & \text{if } x > 1 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
G(x) &= \begin{cases} 0 & \text{if } x < 0, \\ \exp\left\{-\left(\mid \log x \mid\right)^{\alpha}\right\} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x \geq 1 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
G(x) &= \begin{cases} 0 & \text{if } x \leq -1, \\ \exp\left\{-\left(\left|\log\left(\mid x \mid\right)\right|\right)^{-\alpha}\right\} & \text{if } -1 < x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
G(x) &= \begin{cases} \exp\left\{-\left(\log \mid x \mid\right)^{\alpha}\right\} & \text{if } x < -1, \\ 1 & \text{if } x \geq -1 \end{cases} \\
&\quad \text{for some } \alpha > 0; \\
G(x) &= \Phi_1(x), \qquad -\infty < x < \infty; \\
G(x) &= \Psi_1(x), \qquad -\infty < x < \infty.
\end{aligned}$$

Necessary and sufficient conditions for F to belong to the domain of attraction of a p-max stable law are given in Mohan and Subramanya (1991) and Mohan and Ravi (1993).

Sweeting (1985) provides a variant of (2) that seeks norming constants $a_n > 0$ and b_n such that

$$f_n(x) = na_n f(a_n x + b_n) F^{n-1}(a_n x + b_n) \to g(x) \tag{5}$$

as $n \to \infty$, where f_n is the pdf of $(M_n - b_n)/a_n$ and g is the derivative of G. Letting $a(t) = F^{\leftarrow}(1-1/t)$ and $b(t) = tf(a(t))$, the following necessary and sufficient conditions are derived for (5) for distributions in the domain of attraction of the three types distributions:

$$\begin{aligned} I &: \ b(t) \text{ is slowly varying, i.e. } b(tx)/b(t) \to 1 \text{ as } t \to \infty, \\ II &: \ w(F) = \infty \text{ and } \lim_{t\to\infty} a(t)b(t) = \alpha, \\ III &: \ w(F) < \infty \text{ and } \lim_{t\to\infty} \{w(F) - a(t)\} b(t) = \alpha. \end{aligned}$$

In related developments, Pickands (1968) derives conditions under which the various moments of $(M_n - b_n)/a_n$ converge to the corresponding moments of G. It is proved that this is indeed true for all dfs $F \in D(G)$ provided that the moments are finite for sufficiently large n. Pickands (1986) gives necessary and sufficient conditions for the first or second derivatives of the left-hand side of (1) to converge to the corresponding derivative of G. He shows that the second derivatives converge if and only if

$$\lim_{x\uparrow w(F)} \frac{d}{dx}\frac{1-F(x)}{f(x)} = c$$

for some constant $c \in \Re$.

Another extension of (2) is to consider the limiting distribution of $\max(h(X_1), \ldots, h(X_n))$, after suitable normalization, when h is a continuous function defined on some domain. Dorea (1987) provides sufficient conditions for the existence of the limiting distribution as well as a characterization of the limiting distribution.

Recently Nasri-Roudsari (1996) extended (2) for generalized order statistics. Generalized order statistics have been defined by Kamps (1955) as follows: let $k > 0$, $m \in \Re$ be parameters such that $\gamma_r = k + (m+1)(n-r) > 0$ for all $r = 1, \ldots, n-1$, and let $\tilde{m} = (m, \ldots, m)$, a vector of length $n-1$, if $n \geq 2$, $\tilde{m} \in \Re$ arbitrary, if $n = 1$. If the random variables $U_1, \ldots, U_n$ possess a joint probability density function (pdf) of the form

$$k\left(\prod_{i=1}^{n-1}\gamma_i\right)\left(\prod_{i=1}^{n-1}(1-u_i)^m\right)(1-u_n)^{k-1}$$

on the cone $0 \leq u_1 \leq \cdots \leq u_n \leq 1$ of $\Re^n$, then they are called generalized order statistics. Nasri-Roudsari's extension of (2) is that U_n approaches one of the following three dfs after suitable normalization:

$$\begin{aligned} I &: \ \Lambda(x) = \frac{1}{\Gamma\left(\frac{k}{m+1}\right)}\Gamma\left(\frac{k}{m+1}, \exp\{-(m+1)x\}\right), \qquad x \in \Re; \\ II &: \ \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{\Gamma\left(\frac{k}{m+1}\right)}\Gamma\left(\frac{k}{m+1}, x^{-(m+1)\alpha}\right) & \text{if } x \geq 0 \end{cases} \\ &\quad \text{for some } \alpha > 0; \\ III &: \ \Psi_\alpha(x) = \begin{cases} \frac{1}{\Gamma\left(\frac{k}{m+1}\right)}\Gamma\left(\frac{k}{m+1}, (-x)^{(m+1)\alpha}\right) & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \\ &\quad \text{for some } \alpha > 0, \end{aligned} \tag{6}$$

which reduce in the particular case $m = 0$ and $k = 1$ to (2). Here, $\Gamma(a, x)$ denotes the incomplete gamma function

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} \exp(-t) dt,$$

for $a > 0$, $x \geq 0$. Further work by Nasri-Roudsari and Cramer (1999) derives the rate of convergence corresponding to (6).

In practical applications of the limit law, (1), we assume that n is so large that

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) \quad \approx \quad G(x),$$

which implies that

$$\Pr\left(M_n \leq x\right) \approx G\left(\frac{x - b_n}{a_n}\right) \quad = \quad G^*(x), \tag{7}$$

where G^* is of the same type as G. Thus, the three types distributions can be fitted directly to a series of observations of M_n.

Practical applications have been wide-ranging. Type I distributions have been applied to fire protection and insurance problems and the prediction of earthquake magnitudes (Ramachandran, 1982), to model extremely high temperatures (Brown and Katz, 1995), and to predict high return levels of wind speeds relevant for the design of civil engineering structures (Naess, 1998). Type II distributions have been applied to estimate probabilities of extreme occurrences in Germany's stock index (Broussard and Booth, 1998) and to predict the behavior of solar proton peak fluxes (Xapson *et al.*, 1998). Type III distributions have been used to model failure strengths of load-sharing systems (Harlow *et al.*, 1983) and window glasses (Behr *et al.*, 1991), for evaluating the magnitude of future earthquakes in the Pacific (Burton and Makropoulos, 1985), in Argentina (Osella *et al.*, 1992), in Japan (Suzuki and Ozaka, 1994) and in the Indian subcontinent (Rao *et al.*, 1997), for partitioning and floorplanning problems (Sastry and Pi, 1991), to predict the diameter of crops for growth and yield modeling purposes (Kuru *et al.*, 1992), for the analysis of corrosion failures of lead-sheathed cables at the Kennedy Space center (Lee, 1992), to predict the occurrence of geomagnetic storms (Silbergleit, 1996), and to estimate the occurrence probability of giant freak waves in the sea area around Japan (Yasuda and Mori, 1997).

3. GENERALIZED EXTREME VALUE (GEV) DISTRIBUTION

The three types of distributions introduced in (2) may be combined into the single distribution with the cdf

$$G(x) \quad = \quad \exp\left\{-\left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-1/\xi}\right\} \tag{8}$$

defined when $1 + \xi(x - \mu)/\sigma > 0$; $\mu \in \Re$, $\sigma > 0$ and $\xi \in \Re$. This distribution is known as the generalized extreme value distribution. We denote it by $\mathrm{GEV}(\mu, \sigma, \xi)$. The range of values of μ, σ and ξ encompasses the distributions in each of the three types, e.g. GEV(1, 1, $1/\alpha$),

$\alpha > 0$, and GEV$(-1, -1, -1/\alpha)$, $\alpha > 0$, are of the same type as Φ_α and Ψ_α, respectively. Thus, the Fréchet and Weibull types correspond to $\xi > 0$ ($\xi = 1/\alpha$) and $\xi < 0$ ($\xi = -1/\alpha$), respectively. The case $\xi = 0$ is interpreted as the limit $\xi \to 0$ and thus $G(x)$ reduces to the Gumbel type:

$$G(x) \quad = \quad \exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \qquad -\infty < x < \infty. \tag{9}$$

Evidently the value of ξ dictates the tail behavior of G, thus we refer to ξ as the shape parameter. We refer to μ and σ as the location and scale parameters, respectively.

The so-called Annual Maximum Method consists of fitting the Generalized Extreme Value distribution to a series of annual maximum data with n taken to be the number of iid events in a year, i.e. apply equation (7) with the right hand side replaced by (8). Estimates of extreme quantiles of the annual maxima are then obtained by inverting equation (8):

$$G\left(x_T\right) = 1 - \frac{1}{T} \Rightarrow x_T = \mu - \frac{\sigma}{\xi}\left[1 - \left\{-\log\left(1 - 1/T\right)\right\}^{-\xi}\right]. \tag{10}$$

In extreme value terminology, x_T is the return level associated with the return period T, and it is common to extrapolate the relationship (10) to obtain estimates of return levels considerably beyond the end of the data to which the model is fitted.

Various methods of estimation for fitting the Generalized Extreme Value distribution have been proposed. These include: least squares estimation (Maritz and Munro, 1967), direct estimation based on order statistics and records (Hill, 1975; Pickands, 1975; Davis and Resnick, 1984; Dekkers *et al.*, 1989; Berred, 1995; Qi, 1998), maximum likelihood estimation (Prescott and Walden, 1980), probability weighted moments (Hosking *et al.*, 1985), minimum risk point estimation (Mukhopadhyay and Ekwo, 1987), best linear unbiased estimation (Balakrishnan and Chan, 1992), Bayes estimation (Ashour and El-Adl, 1980; Lye *et al.*, 1993), method of moments (Christopeit, 1994), robust bootstrap estimation (Seki and Yokoyama, 1996), and minimum distance estimation (Dietrich and Hüsler, 1996). The most serious competitors are maximum likelihood and probability weighted moments. Madsen *et al.* (1997), Dupuis and Field (1998) and Dupuis (1999b) compare and contrast the performance of these methods. There are a number of regular problems associated with ξ: when $\xi < -1$ the maximum likelihood estimates do not exist, when $-1 < \xi < -1/2$ they may have problems, and when $\xi > 1/2$ second and higher moments do not exist. See Smith (1985a, 1987) for details. The most recent method proposed by Castillo and Hadi (1997) circumvents these problems: it provides well defined estimates for all parameter values and performs well compared to any of the existing methods. In any case, experience with data suggests that the condition $-1/2 < \xi < 1/2$ is valid for most applications.

Also several measures and tests have been devised to assess goodness of fit of the distribution (Stephens, 1977; Tsujitani *et al.*, 1980; Kinnison, 1989; Auinger, 1990; Chowdhury *et al.*, 1991; Aly and Shayib, 1992; Fill and Stedinger, 1995; de Waal, 1996; Zempléni, 1996).

It is often of interest to test whether the extreme values of a physical process are distributed according to a Type I extreme value distribution rather than one of Types II or III. This is equivalent to testing whether $\xi = 0$ in the Generalized Extreme Value distribution. Hosking (1984) compares thirteen tests of this hypothesis and finds evidence to suggest that a modified likelihood ratio test and a test due to van Montfort and Otten (1978) give the best overall performance.

Prescott and Walden (1980) derive the Fisher information matrix for the Generalized Extreme Value distribution. For a sample of n iid observations from (8), the elements of the matrix are:

$$
\begin{aligned}
E\left(-\frac{\partial^2 l}{\partial \mu^2}\right) &= \frac{n}{\sigma^2} p, \\
E\left(-\frac{\partial^2 l}{\partial \sigma^2}\right) &= \frac{n}{\sigma^2 \xi^2}\left\{1 - 2\Gamma(2+\xi) + p\right\}, \\
E\left(-\frac{\partial^2 l}{\partial \xi^2}\right) &= \frac{n}{\xi^2}\left\{\frac{\pi^2}{6} + \left(1 - \gamma + \frac{1}{\xi}\right)^2 - \frac{2q}{\xi} + \frac{p}{\xi^2}\right\}, \\
E\left(-\frac{\partial^2 l}{\partial \mu \partial \sigma}\right) &= -\frac{n}{\sigma^2 \xi}\left\{p - \Gamma(2+\xi)\right\}, \\
E\left(-\frac{\partial^2 l}{\partial \mu \partial \xi}\right) &= -\frac{n}{\sigma \xi}\left(q - \frac{p}{\xi}\right), \\
E\left(-\frac{\partial^2 l}{\partial \sigma \partial \xi}\right) &= -\frac{n}{\sigma \xi^2}\left[1 - \gamma + \frac{\{1 - \Gamma(2+\xi)\}}{\xi} - q + \frac{p}{\xi}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
p &= (1+\xi)^2 \Gamma(1+2\xi), \\
q &= \Gamma(2+\xi)\left\{\Omega(1+\xi) + \frac{1}{\xi} + 1\right\}, \\
\Omega(r) &= \frac{\partial \log \Gamma(r)}{\partial r}
\end{aligned}
$$

and $\gamma = 0.5772157\cdots$ is Euler's constant. Escobar and Meeker (1994) provide an algorithm to evaluate these elements.

4. GENERALIZED PARETO (GP) DISTRIBUTION

Balkema and de Haan (1974) and Pickands (1975) show that $F \in D(G)$, where G is GEV(μ, σ ξ) for some μ, σ and ξ, if and only if

$$
\lim_{t \to w(F)} \sup_{0<x<w(F)-t} \left| \frac{1 - F(t+x)}{1 - F(t)} - \left\{1 + \xi \frac{x}{\sigma_*(t)}\right\}^{-1/\xi} \right| = 0. \tag{11}
$$

The second limiting term within the suprema is the survivor function of the Generalized Pareto (GP) distribution and is defined when either $0 < x < \infty$ ($\xi \geq 0$) or $0 < x < -\sigma_*(t)/\xi$ ($\xi < 0$). The case $\xi = 0$ is again interpreted as the limit $\xi \to 0$ and thus we have the exponential distribution with mean $\sigma_*(t)$ as a special case. We again refer to ξ and σ_* as the shape and scale parameters, respectively.

The Generalized Pareto distribution has been studied extensively and many characterizations exist in the literature. Nagaraja (1977, 1988) provides characterizations based on record values. For a random sample $X_1, X_2, \ldots, X_n$ with common df F let $\{X_{L_n}, n \geq 0\}$ denote the sequence of record values, where $L_0 \equiv 1$ and $L_n = \min(i \mid i > L_{n-1}, X_i > X_{L_{n-1}})$, $n \geq 1$. Then Nagaraja (1977) shows that if for some constants p and q

$$
E\left(X_{L_1} \middle| X_{L_0} = x\right) = px + q \qquad \text{a.s.}
$$

then except for change in location and scale

$$F(x) = \begin{cases} 1-(-x)^\theta, & -1<x<0 \quad \text{if } 0<p<1, \\ 1-\exp(-x), & x>0 \quad \text{if } p=1, \\ 1-x^\theta, & x>1 \quad \text{if } p>1, \end{cases}$$

where $\theta = p/(1-p)$. Nagaraja (1988) extends this result for adjacent record values: namely,

$$E\left(X_{L_m} \left| X_{L_{m+1}} = x\right.\right) = px + q \quad \text{a.s.} \qquad \text{and} \qquad E\left(X_{L_{m+1}} \left| X_{L_m} = x\right.\right) = rx + s \quad \text{a.s.}$$

hold for some real p, q, r and s if and only if F is an exponential df. Kotz and Shanbhag (1980) provide interesting characterizations in terms of the expected remaining life function, $\phi(x) = E(X - x \mid X \geq x)$. They implicitly show that ϕ is a nonvanishing polynomial on (α, β), where $F(\alpha) = 0$ and $F(\beta_-) = 1$, if and only if one of the following is true:

- $\alpha > -\infty$, $\beta = \infty$ and F is a shifted exponential with left extremity α;
- $\alpha > -\infty$, $\beta = \infty$ and F is a Pareto with left extremity α;
- $\alpha > -\infty$, $\beta < \infty$ and for all $x \in (\alpha, \beta)$

$$F(x) = 1 - \frac{\phi\left(\alpha_+\right)}{\phi(x)} \exp\left\{-\int_\alpha^x \frac{dy}{\phi(y)}\right\},$$

where ϕ is a polynomial of the form referred to. When ϕ is linear the distribution in the last case reduces to

$$F(x) = 1 - \left(1 - \frac{x-\alpha}{\beta-\alpha}\right)^\gamma$$

for some $\gamma > 0$.

One of the recent characterizations due to Dembińska and Wesolowski (1998) uses the order statistics of X_i, say $Y_1 < Y_2 < \cdots < Y_n$. They show that if for some $k \leq n - r$ and real p, q

$$E\left(|Y_{k+r}|\right) < \infty \quad \text{and} \quad E\left(Y_{k+r} \left| Y_k\right.\right) = pY_k + q$$

hold then only the following three cases are possible: 1) $p = 1$ and F is an exponential df; 2) $p > 1$ and F is a Pareto df; 3) $p < 1$ and F is a power df. Most recently, Asadi *et al.* (2001) have proposed some unified characterizations of the generalized Pareto using a new concept of 'extended neighboring order statistics'. For other results we refer the reader to the two monographs Galambos and Kotz (1978) and Rao and Shanbhag (1994).

The moments of the Generalized Pareto distribution are readily obtained by noting that

$$E\left(1 + \xi \frac{X}{\sigma_*(t)}\right)^r = \frac{1}{1 - r\xi}$$

if $1 - r\xi > 0$. The rth moment exists if $\xi < 1/r$ and thus all moments exist for $-1/2 < \xi < 0$. The mean and the variance are

$$\frac{\sigma_*(t)}{1-\xi} \quad \text{and} \quad \frac{\sigma_*^2(t)}{(1-\xi)^2(1-2\xi)},$$

respectively.

When t is large enough (11) provides the model

$$F(x) \approx 1 - \omega(t) \left\{ 1 + \xi \frac{x-t}{\sigma_*(t)} \right\}^{-1/\xi}, \tag{12}$$

where $\omega(t) = 1 - F(t)$ is the probability of exceeding t and either $t < x < \infty$ ($\xi \geq 0$) or $t < x < t - \sigma_*(t)/\xi$ ($\xi < 0$). We denote (12) by GP($\sigma_*(t)$, ξ, t). This model forms the basis for the approach of modeling iid exceedances over a high threshold and has the advantage of using more of the available data than just the annual maxima. The analogue of the T-year return level for this model is defined as the value x_T such that $F(x_T) = 1 - 1/(n\omega T)$, where n is the number of iid observations in a year. Inverting (12), we get

$$x_T = t - \frac{\sigma_*(t)}{\xi} \left\{ 1 - (n\omega T)^{\xi} \right\}.$$

The maximum likelihood estimate of $\omega(t)$ is the empirical proportion of exceedances over the threshold t. The maximum likelihood estimates of ξ and $\sigma_*(t)$ exist in large samples provided that $\xi > -1$ and they are asymptotically normal and efficient if $\xi > -1/2$. For an account on how the non-regular cases could be treated see Walshaw (1993) and for an algorithm for computing the maximum likelihood estimates see Grimshaw (1993).

Davison and Smith (1990) give details of statistical inference for (12). For instance, the Fisher information matrix for n iid observations is the inverse of

$$\frac{1+\xi}{n} \left(\begin{array}{cc} 2\sigma_*^2 & \sigma_* \\ \sigma_* & 1+\xi \end{array} \right).$$

Davison and Smith also provide a graphical diagnostic, now known as the *residual life plot*, for selecting a sufficiently large value for the threshold t. For other diagnostics for threshold selection see Nadarajah (1994) and Dupuis (1999a).

Some practical applications of (12) include the estimation of the finite limit of human lifespan (Zelterman, 1992; Aarssen and De Haan, 1994), the modeling of high concentrations in short-range atmospheric dispersion (Mole *et al.*, 1995), and the estimation of flood return levels for homogeneous regions in New Brunswick, Canada (ElJabi *et al.*, 1998).

5. JOINT DISTRIBUTION OF THE R-LARGEST ORDER STATISTICS

Suppose $X_1, X_2, \ldots$ are iid random variables with common df $F \in D(G)$, where G is GEV(μ, σ ξ) for some μ, σ and ξ. Let $M_n^{(i)}$ denote the ith largest of the first n random variables, $i = 1, \ldots, r$. Then, by arguments in Weissman (1978) or by Leadbetter *et al.* (1983, Chapter 2), the limiting joint df of the r largest order statistics for $x_1 \geq x_2 \geq \cdots \geq x_r$ is:

$$\lim_{n \to \infty} \Pr \left\{ \frac{M_n^{(1)} - b_n}{a_n} < x_1, \frac{M_n^{(2)} - b_n}{a_n} < x_2, \ldots, \frac{M_n^{(r)} - b_n}{a_n} < x_r \right\}$$
$$= \sum_{s_1=0}^{1} \sum_{s_2=0}^{2-s_1} \cdots \sum_{s_{r-1}=0}^{r-1-s_1-\cdots-s_{r-2}} \frac{(\gamma_2 - \gamma_1)^{s_1}}{s_1!} \cdots \frac{(\gamma_r - \gamma_{r-1})^{s_{r-1}}}{s_{r-1}!} \exp(-\gamma_r). \tag{13}$$

Here, $\gamma_i = -\log \mathrm{GEV}(x_i; 0, 1, \xi)$, and a_n, b_n are the same norming constants as in (1). Dziubdziela (1978), Deheuvels (1986, 1989) and Falk (1989) provide results on the rate of convergence of (13). Serfozo (1982) identifies necessary and sufficient conditions for the convergence in (13) to hold. Goldie and Maller (1996) provide characterizations of the asymptotic properties,

$$\limsup_{n\to\infty} \left(M_n^{(r)} - M_n^{(r+s)} \right) \leq c \qquad \text{a.s.,}$$

for some finite constant c, which tell us, in various ways, how quickly the sequence of maxima increase. These characterizations take the form of integral conditions on the tail of F. The related problem of the joint distribution of the $(n-r)$ smallest order statistics is considered in Finner and Roters (1994).

Hall (1978) gives canonical representations for a sequence of random variables $\{Z_i\}$ having (13) as finite-dimensional distributions. Let γ denote Euler's constant and let Z_k be independent exponential random variables with mean 1. Then, the representations are that:

$$Z_i =^d \sum_{k=i}^{\infty} \frac{Z_k - 1}{k} + \gamma - \sum_{k=1}^{i-1} \frac{1}{k}, \qquad i \geq 1$$

if X_1 is in the domain of attraction of the Type I distribution;

$$Z_i =^d \exp\left[\alpha^{-1} \left\{ \sum_{k=i}^{\infty} \frac{Z_k - 1}{k} + \gamma - \sum_{k=1}^{i-1} \frac{1}{k} \right\} \right], \qquad i \geq 1$$

if X_1 is in the domain of attraction of the Type II distribution; and,

$$Z_i =^d \exp\left[-\alpha^{-1} \left\{ \sum_{k=i}^{\infty} \frac{Z_k - 1}{k} + \gamma - \sum_{k=1}^{i-1} \frac{1}{k} \right\} \right], \qquad i \geq 1$$

if X_1 is in the domain of attraction of the Type III distribution. Hall also gives limit theorems about the behavior of Z_i as $i \to \infty$.

Practical applications of (13) proceed by assuming that n is sufficiently large for the limit law to hold. Following the argument that led to equation (7), we can express the joint pdf of $(M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(r)})$ in the generalized form:

$$\begin{aligned} f(x_1, x_2, \ldots, x_r) &= \sigma^{-r} \exp\left\{ -\left(1 + \xi \frac{x_r - \mu}{\sigma}\right)^{-1/\xi} \right. \\ &\qquad \left. - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{r} \log\left(1 + \xi \frac{x_i - \mu}{\sigma}\right) \right\} \end{aligned} \tag{14}$$

valid for $x_1 \geq x_2 \geq \cdots \geq x_r$ such that $1 + \xi(x_i - \mu)/\sigma > 0$, $i = 1, 2, \ldots, r$.

Smith (1986), Singh (1987) and Tawn (1988) perform maximum likelihood estimation of (14) and provide algebraic expressions for the associated Fisher information matrix. Dupuis (1997) considers robust estimation. This allows for identification of observations which are not consistent with (14) and an assessment of the validity of the model.

For the asymptotic approximation of (13) to be valid, r has to be small by comparison with n. Actually, as r increases the rate of convergence to the limiting joint distribution decreases sharply. The choice of r is therefore crucial. Wang (1995) proposes a method for selecting r based on a suitable goodness-of-fit statistic. See also Zelterman (1993), Umbach and Ali (1996) and Drees and Kaufmann (1998).

Some practical applications of (14) include the prediction of the extreme pit depths into the future and over large space of exposed metal (Scarf *et al.*, 1992), forest inventory problems (Pierrat *et al.*, 1995), estimation of the extreme sea-level distribution in the Aegean and Ionian Seas (Tsimplis and Blackman, 1997), the prediction of the extreme hurricane wind speeds at locations on the Gulf and Atlantic coasts of the United States (Casson and Coles, 1998) and the prediction of the maximum size of random spheres in Wicksell's corpuscle problem (Takahashi and Sibuya, 1998).

6. APPLICATION TO FLORIDIAN RAINFALL DATA

The first application of extreme value distributions to model rainfall data from Florida was provided by Nadarajah (2003). The data analyzed consisted of annual maximum daily rainfall for the years from 1901 to 2003 for the following fourteen locations in West Central Florida: Clermont, Brooksville, Orlando, Bartow, Avon Park, Arcadia, Kissimmee, Inverness, Plant City, Tarpon Springs, Tampa Intl Airport, St Leo, Gainesville and Ocala. Figure 1 shows how the annual maximum daily rainfall has varied from 1901 to 2002 for two of the fourteen locations (Orlando and Plant City). The variation for both locations appears to exhibit some kind of non-stationarity.

The basic model fitted was (8) with μ, σ, and ξ constant (to be referred to as Model 1). Because of the non-stationarity in Figure 1, Nadarajah (2005) also tried the following variations of model 1:

$$\text{Model 2:} \quad \mu = a + b \times (\text{Year} - t_0 + 1), \quad \sigma = \text{const}, \quad \xi = \text{const},$$

a four parameter model with μ allowed to vary linearly with respect to time; and,

$$\text{Model 3:} \quad \mu = a + b \times (\text{Year} - t_0 + 1) + c \times (\text{Year} - t_0 + 1)^2, \quad \sigma = \text{const}, \quad \xi = \text{const},$$

a five parameter model with μ allowed to vary quadratically with respect to time (where t_0 denotes the year the records started). Higher order polynomials are often better at describing a data set, but their projections into the future tend to fluctuate too wildly – and in the case of variability, they either shrink too quickly or expand too quickly. This was often found to be the case with fitting cubic or higher order polynomials; thus, those models were not considered.

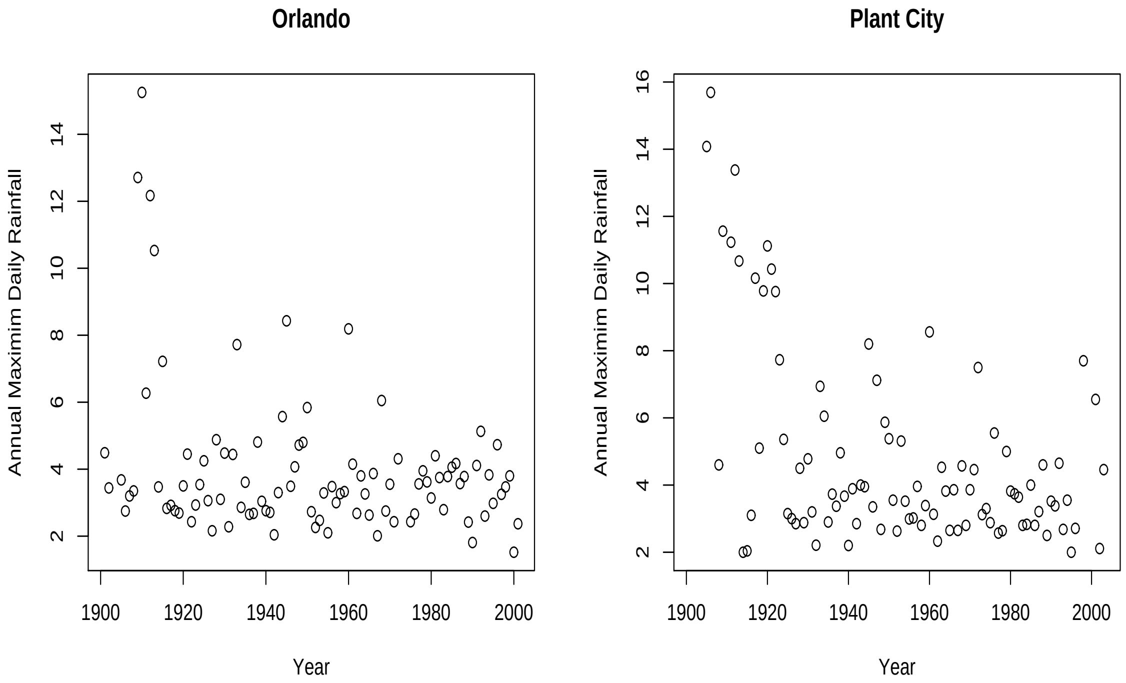

Figure 1. Annual maxima daily rainfall for Orlando (1901–2001) and Plant City (1901–2003).

Models 1 to 3 were fitted to annual maxima daily rainfall from each of the fourteen locations by the method of maximum likelihood. The standard likelihood ratio test was used to discriminate between the models (Wald, 1943). The results showed evidence of non-stationarity for eight of the fourteen locations. Both Orlando and Bartow exhibited a downward linear trend for extreme rainfall. Both Clermont and Kissimmee exhibited a quadratic trend of concave-shape, where extreme rainfall initially decreased before increasing. The change-over appeared to occur between 1940 and 1960. A quadratic trend of convex-shape (where extreme rainfall initially increased before decreasing) was noted for Inverness, Plant City, Tarpon Springs and Tampa International Airport. The change-over again appeared to occur between 1940 and 1960.

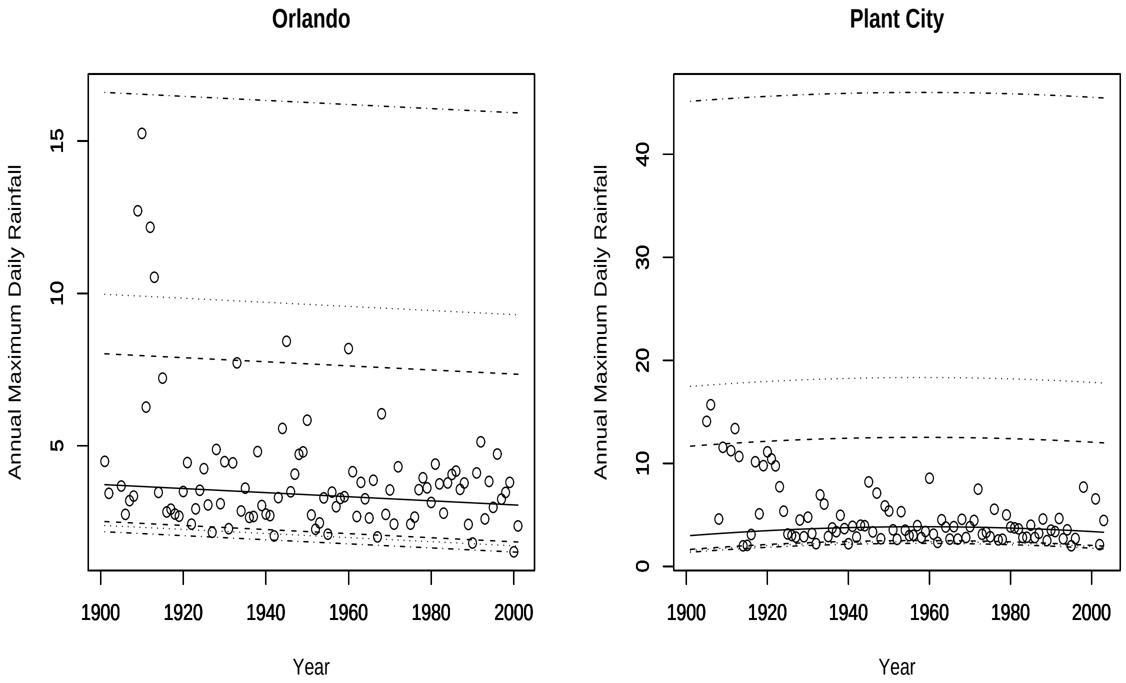

Figure 2. Best fitting models for Orlando (1901–2001) and Plant City (1901–2003).

The graphical displays of the fits for Orlando and Plant City are shown in Figure 2. Each figure shows the original data of annual maximum daily rainfall superimposed with seven different curves. The solid curve is the estimate of the median of the extreme rainfall given by

$$\widehat{\mu} - \frac{\widehat{\sigma}}{\widehat{\xi}} \left\{ 1 - (\log 2)^{-\widehat{\xi}} \right\}$$

while the dotted and dashed curves are the 90%, 95% and 99% confidence intervals given by

$$\left[\widehat{\mu} - \frac{\widehat{\sigma}}{\widehat{\xi}} \left\{ 1 - (\log 20)^{-\widehat{\xi}} \right\}, \widehat{\mu} - \frac{\widehat{\sigma}}{\widehat{\xi}} \left\{ 1 - (-\log 0.95)^{-\widehat{\xi}} \right\} \right],$$

$$\left[\widehat{\mu} - \frac{\widehat{\sigma}}{\widehat{\xi}} \left\{ 1 - (\log 40)^{-\widehat{\xi}} \right\}, \widehat{\mu} - \frac{\widehat{\sigma}}{\widehat{\xi}} \left\{ 1 - (-\log 0.975)^{-\widehat{\xi}} \right\} \right]$$

and

$$\left[\widehat{\mu} - \frac{\widehat{\sigma}}{\widehat{\xi}} \left\{ 1 - (\log 200)^{-\widehat{\xi}} \right\}, \widehat{\mu} - \frac{\widehat{\sigma}}{\widehat{\xi}} \left\{ 1 - (-\log 0.995)^{-\widehat{\xi}} \right\} \right],$$

respectively. These curves can be easily projected beyond the range of the data to provide very useful predictions. The tables below provide predictions for the years 2010, 2020, 2050 and 2100.

Table 1. Predictions of extreme rainfall for Orlando.

Year	Predictions with confidence intervals			
	Median	90% CI	95% CI	99% CI
2010	3.0	(1.8, 7.3)	(1.7, 9.2)	(1.4, 15.9)
2020	2.9	(1.7, 7.2)	(1.6, 9.2)	(1.4, 15.8)
2050	2.7	(1.5, 7.0)	(1.4, 9.0)	(1.2, 15.6)
2100	2.4	(1.2, 6.7)	(1.0, 8.6)	(0.8, 15.3)

Year	Predictions with confidence intervals			
	Median	90% CI	95% CI	99% CI
2010	3.1	(1.8, 11.8)	(1.7, 17.6)	(1.5, 45.3)
2020	2.8	(1.5, 11.5)	(1.4, 17.3)	(1.2, 45.0)
2050	1.6	(0.3, 10.3)	(0.2, 16.1)	(0.0, 43.8)
2100	-1.5	(-2.8, 7.2)	(-2.9, 13.0)	(-3.1, 40.7)

The negative values show the weakness of the models in extrapolating too far into the future. The upper limits of the confidence intervals provide an indication of the probable maximums of extreme rainfall. For example, it is almost certain that the annual maximum daily rainfall in Orlando cannot exceed 15.3 inches in the year 2100.

7. APPLICATION TO NEW ZEALAND WIND DATA

Little has been published on extreme values of New Zealand wind data. Two references that we are aware of are: De Lisle (1965) and Revfeim and Hessell (1984), the former provided an analysis of maximum wind gusts for thirty three stations in New Zealand while the latter suggested an alternative extreme value model with 'physically meaningful parameters' and illustrated its use for wind gust data from four stations around New Zealand.

Withers and Nadarajah (2003) conducted one of the first studies which explored New Zealand wind data for trends. Trends in wind data could be caused by a variety of reasons such as climate change, multidecadal natural climate fluctuations, site movement, site exposure and changes in observational procedure. The data consisted of annual maximum of the daily windrun in km for the following twenty locations in New Zealand: Rarotonga Airport, Leigh, Woodhill Forest, Auckland at Oratia, Waihi, Hamilton at Ruakura, Whatawhata, Rotoehu Forest, Gisborne at Manutuke, Wanganui at Cooks Garden, Palmerston North, Pahiatua at Mangamutu, Wallaceville, Wellington at Kelburn Grassmere Salt Works, Lincoln, Mt Cook at The Hermitage, Winchmore, Waimate and Winton.

The models fitted allowed both the mean and the variability of wind extremes to change with time. Mathematically, if $x_N = N/(n+1)$ denotes the standardized time variable then the model chosen for the annual maximum windrun, Y_N, for years $N = 1, \ldots, n$ is

$$Y_N = \mu_N(\boldsymbol{\alpha}) + \sigma_N(\boldsymbol{\beta})\,\epsilon_N, \tag{15}$$

where

$$\mu_N(\boldsymbol{\alpha}) = \sum_{i=0}^{m} \alpha_i x_N^i$$

and

$$\sigma_N(\boldsymbol{\beta}) = \exp\left(\sum_{i=0}^{m} \beta_i x_N^i\right),$$

giving a model which we shall denote by L(m). Here, $\mu_N(\boldsymbol{\alpha})$, the mean of the process, and $\sigma_N(\boldsymbol{\beta})$, the variability or scale of the process, have been parameterized polynomially with time as polynomial powers have the appeal of simplicity and flexibility to describe a wide variety of behavior – including effects of climate change, site exposure and changes in observational procedure such as instrumental improvements. The case $m = 1$ corresponds to linear trends and $m = 2$ to quadratic trends. The case $m = 0$ corresponds to no trends. The noise or residuals $\epsilon_1, \ldots, \epsilon_n$ are assumed to be independent with the cdf (8).

For each of the twenty data sets (15) was fitted by the method of maximum likelihood for $m = 0, 1, 2$. Withers and Nadarajah (2003) then applied the likelihood ratio test between successive models to discriminate the model giving the best fit. With regard to change with respect to time, there was evidence of significant trend for eight of the twenty locations considered. The station names were: Woodhill Forest, Whatawhata, Rotoehu Forest, Palmerston North, Pahiatua at Mangamutu, Wellington at Kelburn, Winchmore and Winton. The predominant pattern of change was a decrease in mean and an increase in variability. For Woodhill Forest, Whatawhata, Pahiatua at Mangamutu, Wellington at Kelburn, Winchmore and Winton the linear model, L(1), gave the best fit. For Rotoehu Forest and Palmerston North the quadratic model, L(2), gave the best fit. These trend patterns must, however, be treated with caution because of possible effects of change in site, site exposure and observational practices.

To give a simple quantitative impression of the trend patterns Withers and Nadarajah (2003) estimated

- the slope, $100\alpha_1/(n+1)$, representing the magnitude of change in mean annual maximum windrun per century;
- and, the percentage of change in the median of annual maximum windrun over the period of records

for the linear model, L(1), for all eight locations. The estimates and associated standard errors are given in the table below. There are six negative trends and two positive trends.

Table 2. Slopes and percentage changes in maximum windrun.

Station Name	Slope (km/century) and Standard Error	Change in Median of Annual Maximum Windrun (%)
Woodhill Forest	-904 (162)	-41
Whatawhata	-694 (147)	-27
Rotoehu Forest	-400 (136)	-26
Palmerston North	+327 (231)	+20
Pahiatua at Mangamutu	+306 (108)	+24
Wellington at Kelburn	-320 (53)	-21
Winchmore	-543 (99)	-28
Winton	-692 (190)	-16

A further point concerned estimates of the shape parameter ξ under the model, L(0). When these estimates were tested for significance, Withers and Nadarajah (2003) found that five of them were significantly less than zero. Thus, windrun maxima must have a statistical upper bound for these five stations (cf. equation (8)). The station names as well as estimates of the corresponding upper bound with their standard errors are given in the table below. Interestingly, except for Winton, all of them are located in the North Island. The geographical distinction between the North and South Islands could explain this.

Table 3. Upper bounds for maximum daily windrun.

Station Name	Upper Bound (km) and Standard Error
Leigh	1388 (76)
Woodhill Forest	969 (50)
Palmerston North	1001 (35)
Wellington at Kelburn	1522 (236)
Winton	1131 (116)

8. APPLICATION TO SOUTH KOREAN RAINFALL DATA

The only work known to us on extremes of Korean rainfall are that of Park *et al.* (2001) and Park and Jung (2002). Park *et al.* (2001) modeled the summer extreme rainfall data (time series of annual maximum of daily and 2-day precipitation) at 61 gauging stations over South Korea by using the Wakeby distribution with the method of L-moments estimates. Park and Jung (2002) modeled the same data using the Kappa distribution with the maximum likelihood estimates. The main drawback with these approaches is the use of distributions which are not extreme value distributions. Theoretically, there is no justification to model annual maximum daily rainfall by either the Wakeby or the Kappa distribution (although, in practice, they may provide a reasonable fit).

Nadarajah and Choi (2003) provided the first application of extreme value distributions to model rainfall data from South Korea. The data consisted of annual maximum daily rainfall for the years from 1961 to 2001 for five locations. The locations – Seoul in the West, Gangneung in the East, Busan in the South-East, Gwangju in the South-West and Chupungryong in the middle – were chosen carefully to give a good geographical representation of the country.

The basic model fitted was (8) with μ, σ and ξ constant. Sometimes the Gumbel distribution gives as good a fit as (8) for rainfall data, so (9) was also fitted with μ and σ constant. In fact, the Gumbel distribution appeared to give the best description for the five locations except for Gangneung. Nadarajah and Choi (2003) also tried variations of the models for possible trends of with respect to time, but none of them provided a significant fit.

The table below gives estimates of the return level x_T for the best fitting models for $T = 10$, 50 and 100 years. The estimates and the associated 95% confidence intervals were computed using (10) and the profile likelihood method (see, for example, Cox and Hinkley (1974)). For the case (annotated by (—)) confidence intervals are not given because of regularity problems to do with the inversion of the likelihood (Prescott and Walden, 1980).

Table 4. Return level estimates with confidence intervals.

Location	Return level x_T (95% confidence interval)		
	$T = 10$	$T = 50$	$T = 100$
Seoul	240.9 (208.8, 283.5)	323.8 (275.1, 389.9)	359.5 (302.9, 435.2)
Busan	224.0 (194.0, 263.6)	301.5 (255.8, 362.9)	331.8 (281.9, 405.1)
Gwangju	185.5 (162.7, 215.6)	245.9 (211.5, 292.3)	275.1 (232.0, 325.2)
Gangneung	227.7 (185.2, 351.7)	317.9 (—)	374.5 (291.1, 982.5)
Chupungryong	152.3 (134.0, 176.8)	200.2 (172.8, 238.2)	220.7 (189.0, 264.3)

These estimates are very consistent with those given by Park and Jung (2002, page 62). Among the locations considered, the capital Seoul in the West appears to be associated with the highest return levels. This finding is again consistent with Park and Jung (2002), in which the highest 100-year return levels were found in the mid-western part of the Korean peninsula. The second highest return levels appear to be shared by Gangneung and Busan (no significant difference between these two locations). The inland rural area of Chupungryong has the lowest return levels.

9. APPLICATION TO TAIWANESE FLOOD DATA

Flooding in Taiwan is a common phenomenon and often it causes considerable damage. For example, on August of 1997 heavy floods and destructive landslides in northern Taiwan caused millions of dollars in damage and economic losses, and killed over 40 people. Agriculture has been the hardest hit sector. In spite of this, there has been no work concerning extreme values of flood data from Taiwan. As a matter of fact, there has been little concerned with extreme values of any kind of climate data from Taiwan–the only piece of work that we are aware of is Yim *et al.* (1999) where extreme value distributions are applied to study the wind climate at Taichung harbor.

Nadarajah and Shiau (2005) provided the first application of extreme value distributions to flood data from Taiwan. The daily streamflow data from the Pachang River located in southern Taiwan was used. Thirty-nine yearly daily streamflow records, from 1961 to 1999, were employed to investigate the extreme floods. Flood events were defined as daily streamflow exceeding a given threshold, which was selected as 100 cms in this study. The following four properties were used to characterize flood events:

- flood peak was defined as the maximum daily flow during the flood period,
- flood volume was defined as the cumulative flow volume during the flood period,
- flood duration was defined as the time period when the flow exceeds the threshold,
- and the time of peak was defined as the number of days counting 1/1/1961 to the day of flood peak.

Fifty flood events were abstracted from the daily streamflow data of the Pachang River.

Goel *et al.* (1998) and Yue *et al.* (1999) suggested that flood flows are not only determined by their peak and volume, but also by other characteristics such as flood duration and the time of flood peak. Thus, it is natural to ask how the extreme values of flood volume and flood peak vary with respect to the duration and time. To investigate this, Nadarajah and Shiau (2005) considered a number of variants of the Gumbel distribution (9) with the location parameter μ expressed as functions of flood duration (D) and the time of flood peak (T). These included a three parameter model with μ allowed to vary linearly with flood duration:

$$\text{Model 1:} \quad \mu = a + b \times D, \quad \sigma = \text{const};$$

a three parameter model with μ allowed to vary linearly with the time of peak:

$$\text{Model 2:} \quad \mu = a + b \times T, \quad \sigma = \text{const};$$

and, a four parameter model with μ allowed to vary linearly with both flood duration and the time of peak:

$$\text{Model 3:} \quad \mu = a + b \times D + c \times T, \quad \sigma = \text{const}.$$

The standard likelihood ratio test was used to determine whether the trends described by these models were significant or not (Wald, 1943). Results showed that flood volume has a highly significant upward trend with respect to flood duration corresponding to an increase of 180.3 cms per unit change in flood duration and that flood peak has a significant downward trend with respect to flood duration corresponding to a decrease of 40.7 cms per unit change in flood duration.

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