

EXAMPLE: VaR is not Sub-additive

The sub-additive property requires that a risk measure $\rho[X]$ is sub-additive if:

$$\rho[A + B] \leq \rho[A] + \rho[B]$$

$$\rho[A + B + C] \leq \rho[A] + \rho[B] + \rho[C]$$

etc.

Let there be three securities A, B and C – zero coupon bonds selling at par value (100) are good – for each security there are three possible future outcomes for the price $\{X = 110, Y = 100, Z = 10\}$ with Probabilities $\text{Prob}[X] = 0.9$ (90%) $\text{Prob}[Y] = .095$ (9.5%) $\text{Prob}[Z] = .005$ (0.5%)

The 99% $\text{VaR}[A] = 100 - 100$ (Initial Price) = 0 = 99% $\text{VaR}[B] = 99\%$ $\text{VaR}[C]$

$$\text{VaR}[A] + \text{VaR}[B] + \text{VaR}[C] = 0$$

Consider the outcomes for the portfolio with three securities

$$\text{Prob}[X/A] * \text{Prob}[X/B] * \text{Prob}[X/C] = (.9) * (.9) * (.9) = .729$$

$$\text{Prob}[X/A] * \text{Prob}[X/B] * \text{Prob}[Y/C]$$

$$\text{Prob}[X/A] * \text{Prob}[Y/B] * \text{Prob}[X/C] = (.9) * (.9) * (.095) = .07695$$

$$\text{Prob}[Y/A] * \text{Prob}[X/B] * \text{Prob}[X/C] = .23085 \quad (3 \text{ cases of } 2X \text{ with } Y)$$

$$\text{Prob}[Y/A] * \text{Prob}[Y/B] * \text{Prob}[X/C]$$

$$\text{Prob}[Y/A] * \text{Prob}[X/B] * \text{Prob}[Y/C] = .02436$$

$$\text{Prob}[X/A] * \text{Prob}[Y/B] * \text{Prob}[Y/C] = (.9) * (.095) * (.095) = .0081225$$

$$\text{Prob}[Y/A] * \text{Prob}[Y/B] * \text{Prob}[Y/C] = \text{VaR}[0] = .095 * .095 * .095 = .008574$$

$$.729 + .23085 + .008574 = .9607$$

==> ALL RETURNS BELOW THIS AMOUNT HAVE A $\text{VaR} > 0$ (Remember VaR is a positive number when there is a loss)

Therefore, VaR is not sub-additive