MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 9

Show each of the following is a valid copula.

1. the independence copula defined by $C(u_1, u_2) = u_1 u_2$.
2. the copula defined by $C(u_1, u_2) = \min(u_1, u_2)$.
3. the copula defined by
$$C(u_1, u_2) = u_1 u_2 \exp[-\theta \log u_1 \log u_2]$$
for $0 < \theta \leq 1$.
4. the Farlie-Gumbel-Morgenstern copula defined by
$$C(u_1, u_2) = u_1 u_2 [1 + \phi(1 - u_1)(1 - u_2)]$$
for $-1 \leq \phi \leq 1$.
5. Burr copula defined by
$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right]^{-\alpha}$$
for $\alpha > 0$.
6. Marshall and Olkin's copula defined by
$$C(u_1, u_2) = \begin{cases} u_1^{1-\alpha} u_2, & \text{if } u_1^\alpha \geq u_2^\beta, \\ u_1 u_2^{1-\beta}, & \text{if } u_1^\alpha < u_2^\beta \end{cases}$$
for $0 \leq \alpha, \beta \leq 1$.

1) Show that $C^{I}(u_1, v_2) = u_1 v_2$ is a copula.

(i) $C^{I}(u, 0) = u \cdot 0 = 0$ ✓

(ii) $C^{I}(0, u) = 0 \cdot u = 0$ ✓

(iii) $C^{I}(1, u) = 1 \cdot u = u$ ✓

(iv) $C^{I}(u, 1) = u \cdot 1 = u$ ✓

(v) $\frac{\partial}{\partial u_1} C^{I}(u_1, v_2) = u_2 \geq 0$ ✓

(vi) $\frac{\partial}{\partial u_2} C^{I}(u_1, u_2) = u_1 \geq 0$ ✓

Hence, C^{I} is a copula.

2) Show that $C^{1}(u_1, u_2) = \min(u_1, u_2)$ is a copula.

(i) $C^{1}(u, 0) = \min(u, 0) = 0$ ✓

(ii) $C^{1}(0, u) = \min(0, u) = 0$ ✓

(iii) $C^{1}(1, u) = \min(1, u) = u$ ✓

(iv) $C^{1}(u, 1) = \min(u, 1) = u$ ✓

(v) $$\frac{\partial}{\partial u_1} C^{1}(u_1, u_2) = \frac{\partial}{\partial u_1} \begin{cases} u_1 & \text{if } u_1 \leqslant u_2 \\ u_2 & \text{if } u_1 > u_2 \end{cases}$$

$$= \begin{cases} 1 & \text{if } u_1 \leqslant u_2 \\ 0 & \text{if } u_1 > u_2 \end{cases}$$

$$\geqslant 0 \checkmark$$

(vi) $$\frac{\partial}{\partial u_2} C^{1}(u_1, u_2) = \frac{\partial}{\partial u_2} \begin{cases} u_1 & \text{if } u_1 \leqslant u_2 \\ u_2 & \text{if } u_1 > u_2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } u_1 \leqslant u_2 \\ 1 & \text{if } u_1 > u_2 \end{cases}$$

$$\geqslant 0 \checkmark$$

Hence, C^{1} is a copula.

3) Show that $C(u_1, u_2) = u_1 u_2 e^{-\theta(\log u_1)(\log u_2)}$, $0 < \theta \leq 1$

is a copula.

(i) $C(u, 0) = u \cdot 0 \cdot e^{-\theta(\log u)\cdot(\log 0)}$

$= 0$ ✓

(ii) $C(0, u) = 0 \cdot u \cdot e^{-\theta(\log 0)\cdot(\log u)}$

$= 0$ ✓

(iii) $C(1, u) = 1 \cdot u \cdot e^{-\theta(\log 1)\cdot(\log u)}$

$= u$ ✓

(iv) $C(u, 1) = u \cdot 1 \cdot e^{-\theta(\log u)\cdot(\log 1)}$

$= u$ ✓

(v) $\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 e^{-\theta(\log u_1)(\log u_2)} + u_1 u_2 e^{-\theta(\log u_1)(\log u_2)}\left(-\frac{\theta}{u_1}\right)(\log u_2)$

$= \underset{\geq 0}{u_2}\, \underset{\geq 0}{\boxed{e^{-\theta(\log u_1)(\log u_2)}}}\, \boxed{\left[1 - \theta \underset{\leq 0}{\log u_2}\right]} \geq 0$

≥ 0 ✓

(vi) $$\frac{\partial}{\partial u_2} C_1^1(u_1, u_2) = \underset{\geq 0}{u_1}\, \underset{\geq 0}{\boxed{e^{-\theta(\log u_1)(\log u_2)}}}\, \boxed{\left[1 - \theta \underset{\leq 0}{\boxed{\log u_1}}\right]} \geq 0$$

$$\geq 0 \checkmark$$

Hence, C_1^1 is a copula.

4) Show that $C(u_1, u_2) = u_1 u_2[1+\phi(1-u_1)(1-u_2)]$, $-1 \leq \phi \leq 1$

is a copula.

(i) $C(u, 0) = u \cdot 0 \cdot [1+\phi \cdot (1-u) \cdot (1-0)]$
$= 0$ ✓

(ii) $C(0, u) = 0 \cdot u \cdot [1+\phi \cdot (1-0) \cdot (1-u)]$
$= 0$ ✓

(iii) $C(1, u) = 1 \cdot u \cdot [1+\phi \cdot (1-1) \cdot (1-u)]$
$= u$ ✓

(iv) $C(u, 1) = u \cdot 1 \cdot [1+\phi \cdot (1-u) \cdot (1-1)]$
$= u$ ✓

(v) $\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2[1+\phi(1-u_1)(1-u_2)]$
$+ u_1 u_2[-\phi(1-u_2)]$

$= \underset{\geq 0}{u_2}[1 + \boxed{\underset{-1 \to 1}{\phi}\,\underset{-1 \to 1}{(1-2u_1)}\,\underset{\geq 0}{(1-u_2)}}]$

$-1 \to 1$

≥ 0 ✓

(vi) $\frac{\partial}{\partial u_2} G'(u_1, u_2) = u_1 \left[1 + \phi (1 - 2u_2)(1 - u_1) \right]$

$u_1 \geq 0$

$\phi: -1 \to 1$, $(1-2u_2): -1 \to 1$, $(1-u_1): 0 \to 1$, $\phi(1-2u_2)(1-u_1): -1 \to 1$

≥ 0 ✓

Hence, G' is a copula.

5) Show that

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1-u_1)^{-\frac{1}{\alpha}} + (1-u_2)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha}$$

is a copula.

(i) $C(u, 0) = u + 0 - 1 + \left[(1-u)^{-\frac{1}{\alpha}} + 1 - 1\right]^{-\alpha}$

$= u - 1 + 1 - u$

$= 0$ ✓

(ii) $C(0, u) = 0 + u - 1 + \left[1 + (1-u)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha}$

$= u - 1 + 1 - u$

$= 0$ ✓

(iii) $C(u, 1) = u + 1 - 1 + \left[(1-u)^{-\frac{1}{\alpha}} + (1-1)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha}$

$= u + \left[(1-u)^{-\frac{1}{\alpha}} + 0^{-\frac{1}{\alpha}} - 1\right]^{-\alpha}$

$= u + \left[(1-u)^{-\frac{1}{\alpha}} + \infty - 1\right]^{-\alpha}$

$= u + (\infty)^{-\alpha}$

$= u + 0$

$= u$ ✓

(iv) $$C(1,u) = 1 + u - 1 + \left[(1-1)^{-\frac{1}{\alpha}} + (1-u)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha}$$

$$= u + \left[0^{-\frac{1}{\alpha}} + (1-u)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha}$$

$$= u + \left[\infty + (1-u)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha}$$

$$= u + (\infty)^{-\alpha}$$

$$= u + 0$$

$$= u \quad \checkmark$$

(v) $$\frac{\partial}{\partial u_1} C(u_1, u_2) = 1 - (1-u_1)^{-\frac{1}{\alpha}-1}\left[(1-u_1)^{-\frac{1}{\alpha}} + (1-u_2)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha-1}$$

$$= 1 - \underbrace{\left[\frac{\underbrace{(1-u_1)^{-\frac{1}{\alpha}}}_{\geq 1}}{\underbrace{(1-u_1)^{-\frac{1}{\alpha}}}_{\geq 1} + \underbrace{\underbrace{(1-u_2)^{-\frac{1}{\alpha}}}_{\geq 1} - 1}_{\geq 0}}\right]^{\alpha+1}}_{\leq 1}$$

$$\geq 0 \quad \checkmark$$

(vi) $\dfrac{\partial}{\partial u_2} C(u_1, u_2)$

$$= 1 - (1-u_2)^{-\frac{1}{\alpha}-1}\left[(1-u_1)^{-\frac{1}{\alpha}} + (1-u_2)^{-\frac{1}{\alpha}} - 1\right]^{-\alpha-1}$$

$$= 1 - \left[\frac{\boxed{(1-u_2)^{-\frac{1}{\alpha}}} \geq 1}{\underbrace{\underbrace{\boxed{(1-u_1)^{-\frac{1}{\alpha}}}}_{\geq 1} + \underbrace{\boxed{(1-u_2)^{-\frac{1}{\alpha}}}}_{\geq 1} - 1}_{\geq 0}}\right]^{\alpha+1}$$

$$\leq 1$$

$$\geq 0 \checkmark$$

Hence, C is a copula.

6) Show that $C_1(u_1, u_2) = \begin{cases} u_1^{1-\alpha} u_2, & u_1^{\alpha} \geq u_2^{\beta} \\ u_1 u_2^{1-\beta}, & u_1^{\alpha} < u_2^{\beta} \end{cases}$

is a copula.

(i) $C_1(u, 0) = u^{1-\alpha} \cdot 0 = 0$ ✓

(ii) $C_1(0, u) = 0 \cdot u^{1-\beta} = 0$ ✓

(iii) $C_1(u, 1) = u \cdot 1^{1-\beta} = u$ ✓

(iv) $C_1(1, u) = 1^{1-\alpha} \cdot u = u$ ✓

(v) $$\frac{\partial}{\partial u_1} C_1(u_1, u_2) = \begin{cases} (1-\alpha) u_1^{-\alpha} u_2 & \text{if } u_1^{\alpha} \geq u_2^{\beta} \\ u_2^{1-\beta} & \text{if } u_1^{\alpha} < u_2^{\beta} \end{cases}$$

≥ 0 ✓

(vi) $$\frac{\partial}{\partial u_2} C_1(u_1, u_2) = \begin{cases} u_1^{1-\alpha} & \text{if } u_1^{\alpha} \geq u_2^{\beta} \\ (1-\beta) u_1 u_2^{-\beta} & \text{if } u_1^{\alpha} < u_2^{\beta} \end{cases}$$

≥ 0 ✓

Hence, C_1 is a copula.