# MATH4/68181: Extreme values and financial risk
# Semester 1
# Problem sheet for Week 8

1. If $x_1, x_2, \ldots, x_n$ is a random sample from Exp $(\lambda)$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

2. If $x_1, x_2, \ldots, x_n$ is a random sample from the power function distribution with pdf $f(x) = ax^{a-1}$, $0 < x < 1$, $a > 0$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

3. If $x_1, x_2, \ldots, x_n$ is a random sample from the normal distribution $N(\mu, \sigma^2)$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

4. If $x_1, x_2, \ldots, x_n$ is a random sample from the log-normal distribution $LN(\mu, \sigma^2)$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

5. If $x_1, x_2, \ldots, x_n$ is a random sample from a distribution with pdf $f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$, $0 < x < \theta_1$, $\theta_1 > 0$, $\theta_2 > 0$ find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

6. If $x_1, x_2, \ldots, x_n$ is a random sample from the uniform $[\mu - \delta, \mu + \delta]$ distribution find the maximum likelihood estimates of $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_p(X)$.

Q1 $X \sim Exp(\lambda)$

We know

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

and

$$ES_p(X) = -\frac{1}{\lambda p}\left\{p\log(1-p) - p - \log(1-p)\right\}$$

The likelihood is

$$L(\lambda) = \left[\prod_{i=1}^{n} \lambda e^{-\lambda x_i}\right]$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

Its log is

$$\log L(\lambda) = n\log\lambda - \lambda\sum_{i=1}^{n} x_i$$

The derivative wrt $\lambda$ is

$$\frac{d\log L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}.$$

The 2nd derivative wrt $\lambda$ is

$$\frac{d^2\log L(\lambda)}{d\lambda^2} = -\frac{n}{\lambda^2} < 0.$$

Hence, $\hat{\lambda} = \frac{1}{\bar{x}}$ is the MLE of $\lambda$.

Hence, the MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = -\bar{x}\log(1-p)$$

and $$\widehat{ES}_p(X) = -\frac{\bar{x}}{p}\left\{p\cdot\log(1-p) - p - \log(1-p)\right\}.$$

**Q2** X has PDF $f(x) = a x^{a-1}$,
$0 < x < 1$.

$$VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}.$$

The likelihood is

$$L(a) = \prod_{i=1}^{n} (a x_i^{a-1})$$

$$= a^n \left(\prod_{i=1}^{n} x_i\right)^{a-1}$$

Its log is

$$\log L(a) = n \log a + (a-1) \sum_{i=1}^{n} \log x_i$$

The derivative wrt a is

$$\frac{d \log L}{d a} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i = 0$$

$$\Rightarrow \hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$$

The 2nd derivative wrt a is

$$\frac{d^2 \log L}{d a^2} = -\frac{n}{a^2} < 0.$$

Hence, $\hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$ is an MLE of a.

Hence, the MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = p^{\frac{1}{\widehat{a}}}$$

and

$$\widehat{ES}_p(X) = \frac{p^{\frac{1}{\widehat{a}}}}{\frac{1}{\widehat{a}} + 1}.$$

Q3 $X \sim N(\mu, \sigma^2)$.

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p)$$

and

$$ES_p(X) = \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt$$

The likelihood function is

$$L(\mu, \sigma) = \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}\right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}.$$

Its log is

$$\log L(\mu, \sigma) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2.$$

The partial derivatives wrt $\mu$ and $\sigma$ are

$$\frac{\partial \log L}{\partial \mu} = \frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$= \frac{1}{\cancel{2}\sigma^2}\left[\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n}\mu\right)\right]$$

$$= \frac{1}{\cancel{2}\sigma^2}\left[\sum_{i=1}^{n} x_i - n\mu\right] = 0 \quad \text{——— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 = 0 \quad -(2)$$

$$(1) \Longrightarrow \quad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}.$$

$$(2) \Rightarrow \quad \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Hence, the MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = \hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)$$

and $$\widehat{ES}_p(X) = \hat{\mu} + \frac{\hat{\sigma}}{p}\int_0^p \Phi^{-1}(t)\,dt.$$

Q4

$$X \sim LN(\mu, \sigma^2)$$

$$F_X(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right).$$

Set $F_X(x) = p$

$$\Rightarrow \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = \exp\left[\mu + \sigma \Phi^{-1}(p)\right]$$

$$\Rightarrow VaR_p(X) = \exp\left[\mu + \sigma \Phi^{-1}(p)\right].$$

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_t(X)\, dt$$

$$= \frac{1}{p} \int_0^p \exp\left[\mu + \sigma \Phi^{-1}(t)\right] dt$$

$$= \frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)}\, dt$$

The MLEs of $\mu$ and $\sigma$ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

and $$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log x_i - \hat{\mu})^2}$$

Hence, the MLEs of VaR and ES are

$$\widehat{VaR}_p(X) = \exp\left[\hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)\right]$$

and

$$\widehat{ES}_p(X) = \frac{e^{\hat{\mu}}}{p} \int_0^p e^{\hat{\sigma}\,\Phi^{-1}(t)}\,dt$$

Q5

X has PDF

$$f(x) = \theta_2 \theta_1^{-\theta_2} x^{\theta_2 - 1},$$
$$0 < x < \theta_1$$

The CDF is

$$F(x) = \int_0^x f(y)\,dy$$
$$= \theta_2 \theta_1^{-\theta_2} \int_0^x y^{\theta_2 - 1}\,dy$$
$$= \theta_2 \theta_1^{-\theta_2} \left[\frac{y^{\theta_2}}{\theta_2}\right]_0^x$$
$$= \left(\frac{x}{\theta_1}\right)^{\theta_2}.$$

Set $F(x) = p$

$$\Rightarrow \left(\frac{x}{\theta_1}\right)^{\theta_2} = p$$
$$\Rightarrow x = \theta_1 p^{\frac{1}{\theta_2}}$$
$$\Rightarrow VaR_p(X) = \theta_1 p^{\frac{1}{\theta_2}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_t(X)\,dt$$
$$= \theta_1 \frac{p^{\frac{1}{\theta_2}}}{\frac{1}{\theta_2} + 1}.$$

The MLEs of $\theta_1$ and $\theta_2$ are

$$\widehat{\theta}_1 = \max(x_1, \ldots, x_n)$$

and

$$\widehat{\theta}_2 = \frac{n}{n \log \widehat{\theta}_1 - \sum_{i=1}^{n} \log x_i}.$$

Hence, the MLEs of VaR and ES are

$$\widehat{VaR_p(X)} = \widehat{\theta}_1\, p^{\frac{1}{\widehat{\theta}_2}}$$

and

$$\widehat{ES_p(X)} = \frac{\widehat{\theta}_1\, p^{\frac{1}{\widehat{\theta}_2}}}{1 + \frac{1}{\widehat{\theta}_2}}.$$