MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 6 and Week 7

Suppose X is a random variable representing a financial loss. Find expressions for Value at Risk and Expected Shortfall if X has the following distributions:

1. the exponential distribution given by the cdf $F(x) = 1 - \exp(-\lambda x)$, $x > 0$.
2. the power function distribution given by the cdf $F(x) = x^a$, $0 < x < 1$.
3. the uniform$[a, b]$ distribution.
4. the Pareto distribution given by the cdf $1 - (K/x)^a$, $x > K$.
5. the standard normal distribution.
6. the log-logistic distribution $\left[1 + (x/a)^{-b}\right]^{-1}$, $x > 0$.
7. the Lomax distribution given by the cdf $1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$, $x > 0$.
8. the Fréchet distribution given by the cdf $\exp\left\{-\left(\frac{\sigma}{x}\right)^{\alpha}\right\}$, $x > 0$.
9. the Weibull distribution given by the cdf $1 - \exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}$, $x > 0$.
10. the generalized Pareto distribution given by the cdf $1 - \left(1 - \frac{cx}{k}\right)^{1/c}$, $x > 0$.
11. the Tukey Lambda distribution given by the quantile function $\frac{p^{\lambda} - (1-p)^{\lambda}}{\lambda}$.
12. the generalized Lambda distribution given by the quantile function $\frac{p^{\beta} - (1-p)^{\gamma}}{\delta}$.
13. Hankin and Lee's distribution given by the quantile function $\frac{Cp^{\alpha}}{(1-p)^{\beta}}$.

7) $F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x > 0.$

Set $F(x) = p$

$\Rightarrow \quad 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$

$\Rightarrow \quad \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$

$\Rightarrow \quad 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$

$\Rightarrow \quad x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$

$\Rightarrow \quad VaR_p(X) = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{\lambda}{p}\int_0^p \left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left[\frac{(1-t)^{1-\frac{1}{\alpha}}}{\left(1-\frac{1}{\alpha}\right)(-1)} - t\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}} - 1}{\frac{1}{\alpha} - 1} - p\right].$$

8)

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}}, \quad x > 0$$

Set $F(x) = p$

$$\Rightarrow e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow \left(\frac{\sigma}{x}\right)^{\alpha} = -\log p$$

$$\Rightarrow \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow x = \sigma(-\log p)^{-\frac{1}{\alpha}}.$$

$$\Rightarrow VaR_p(X) = \sigma(-\log p)^{-\frac{1}{\alpha}}.$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{\sigma}{p}\int_0^p (-\log t)^{-\frac{1}{\alpha}}\,dt$$

$$\boxed{\begin{aligned} &\text{Set } y = -\log t \\ &\Rightarrow t = e^{-y} \\ &\Rightarrow \frac{dt}{dy} = -e^{-y} \end{aligned}}$$

$$= \frac{\sigma}{p}\int_{\infty}^{-\log p} y^{-\frac{1}{\alpha}}(-e^{-y})\,dy$$

$$= \frac{\sigma}{p}\int_{-\log p}^{\infty} y^{-\frac{1}{\alpha}} e^{-y}\,dy$$

$$\boxed{\begin{array}{l}\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \\ \underline{\text{Lower}} \text{ incomplete gamma function}\end{array}}$$

$$= \frac{\sigma}{p} \Gamma\left(1 - \frac{1}{\alpha}, -\log p\right).$$

9) $$F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^{\alpha}}, \quad x > 0$$

Set $F(x) = p$

$$\Rightarrow 1 - e^{-\left(\frac{x}{\sigma}\right)^{\alpha}} = p$$

$$\Rightarrow e^{-\left(\frac{x}{\sigma}\right)^{\alpha}} = 1 - p$$

$$\Rightarrow \left(\frac{x}{\sigma}\right)^{\alpha} = -\log(1-p)$$

$$\Rightarrow x = \sigma\left[-\log(1-p)\right]^{\frac{1}{\alpha}}$$

$$\Rightarrow VaR_p(X) = \sigma\left[-\log(1-p)\right]^{\frac{1}{\alpha}}.$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{\sigma}{p}\int_0^p \left[-\log(1-t)\right]^{\frac{1}{\alpha}}\,dt$$

Set $y = -\log(1-t)$

$$\Rightarrow 1 - t = e^{-y}$$

$$\Rightarrow t = 1 - e^{-y}$$

$$\Rightarrow \frac{dt}{dy} = e^{-y}$$

$$= \frac{\sigma}{p}\int_0^{-\log(1-p)} y^{\frac{1}{\alpha}} e^{-y}\,dy$$

$$\boxed{\begin{array}{l}\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt \\ \underline{\text{Upper}} \text{ incomplete gamma function}\end{array}}$$

$$= \frac{\sigma}{p} \gamma\left(1 + \frac{1}{\alpha}, -\log(1-p) \right).$$

10) $$F(x) = 1 - \left(1 - \frac{cx}{k}\right)^{\frac{1}{c}}$$

Set $F(x) = p$

$$\Rightarrow 1 - \left(1 - \frac{cx}{k}\right)^{\frac{1}{c}} = p$$

$$\Rightarrow \left(1 - \frac{cx}{k}\right)^{\frac{1}{c}} = 1 - p$$

$$\Rightarrow 1 - \frac{cx}{k} = (1-p)^c$$

$$\Rightarrow \frac{cx}{k} = 1 - (1-p)^c$$

$$\Rightarrow x = \frac{k}{c}\left[1 - (1-p)^c\right]$$

$$\Rightarrow VaR_p(X) = \frac{k}{c}\left[1 - (1-p)^c\right].$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{k}{cp}\int_0^p \left[1 - (1-t)^c\right] dt$$

$$= \frac{k}{cp}\left[t - \frac{(1-t)^{c+1}}{(c+1)(-1)}\right]_0^p$$

$$= \frac{k}{cp}\left[p + \frac{(1-p)^{c+1} - 1}{c+1}\right].$$

11)

$F^{-1}(p)$ = quantile function

$$VaR_p(X) = \frac{p^\lambda - (1-p)^\lambda}{\lambda}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p\lambda}\int_0^p \left[t^\lambda - (1-t)^\lambda\right] dt$$

$$= \frac{1}{p\lambda}\left[\frac{t^{\lambda+1}}{\lambda+1} - \frac{(1-t)^{\lambda+1}}{(\lambda+1)(-1)}\right]_0^p$$

$$= \frac{1}{p\lambda}\left[\frac{p^{\lambda+1}}{\lambda+1} + \frac{(1-p)^{\lambda+1} - 1}{\lambda+1}\right].$$

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$$F^{-1}(p) = \frac{p^{\beta} - (1-p)^{\gamma}}{\delta}$$

$$VaR_p(X) = \frac{p^{\beta} - (1-p)^{\gamma}}{\delta}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p\delta}\int_0^p \left[t^{\beta} - (1-t)^{\gamma}\right] dt$$

$$= \frac{1}{p\delta}\left[\frac{t^{\beta+1}}{\beta+1} - \frac{(1-t)^{\gamma+1}}{(\gamma+1)(-1)}\right]_0^p$$

$$= \frac{1}{p\delta}\left[\frac{p^{\beta+1}}{\beta+1} + \frac{(1-p)^{\gamma+1} - 1}{\gamma+1}\right].$$

13) $$F^{-1}(p) = \frac{C_1 p^{\alpha}}{(1-p)^{\beta}}$$

$$VaR_p(X) = \frac{C_1 p^{\alpha}}{(1-p)^{\beta}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{C_1}{p}\int_0^p \frac{t^{\alpha}}{(1-t)^{\beta}}\,dt$$

$$= \frac{C_1}{p}\int_0^p t^{\alpha}(1-t)^{-\beta}\,dt$$

$$\boxed{B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt}$$

Incomplete beta function

$$= \frac{C_1}{p} B_p(1+\alpha, 1-\beta).$$