MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 6 and Week 7

Suppose X is a random variable representing a financial loss. Find expressions for Value at Risk and Expected Shortfall if X has the following distributions:

1. the exponential distribution given by the cdf $F(x) = 1 - \exp(-\lambda x)$, $x > 0$.
2. the power function distribution given by the cdf $F(x) = x^a$, $0 < x < 1$.
3. the uniform$[a, b]$ distribution.
4. the Pareto distribution given by the cdf $1 - (K/x)^a$, $x > K$.
5. the standard normal distribution.
6. the log-logistic distribution $\left[1 + (x/a)^{-b}\right]^{-1}$, $x > 0$.
7. the Lomax distribution given by the cdf $1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$, $x > 0$.
8. the Fréchet distribution given by the cdf $\exp\left\{-\left(\frac{\sigma}{x}\right)^{\alpha}\right\}$, $x > 0$.
9. the Weibull distribution given by the cdf $1 - \exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}$, $x > 0$.
10. the generalized Pareto distribution given by the cdf $1 - \left(1 - \frac{cx}{k}\right)^{1/c}$, $x > 0$.
11. the Tukey Lambda distribution given by the quantile function $\frac{p^{\lambda} - (1-p)^{\lambda}}{\lambda}$.
12. the generalized Lambda distribution given by the quantile function $\frac{p^{\beta} - (1-p)^{\gamma}}{\delta}$.
13. Hankin and Lee's distribution given by the quantile function $\frac{Cp^{\alpha}}{(1-p)^{\beta}}$.

1) Set $F(x) = p$

$$\Rightarrow \quad 1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad x = -\frac{1}{\lambda} \log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda} \log(1-p).$$

$$ES_p(X) = \frac{1}{p} \int_0^p VaR_t(X)\, dt$$

$$= -\frac{1}{\lambda p} \int_0^p \log(1-t)\, dt$$

Int by parts

$$= -\frac{1}{\lambda p} \left\{ \left[t \cdot \log(1-t) \right]_0^p + \int_0^p \frac{t}{1-t}\, dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) + \int_0^p \frac{(t-1)+1}{1-t}\, dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) + \left[-t - \log(1-t) \right]_0^p \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) - p - \log(1-p) \right\}.$$

2)

$$F(x) = x^a$$

Set $F(x) = p$

$$\Rightarrow \quad x^a = p$$

$$\Rightarrow \quad x = p^{\frac{1}{a}}$$

$$\Rightarrow \quad VaR_p(X) = p^{\frac{1}{a}}.$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p t^{\frac{1}{a}}\,dt$$

$$= \frac{1}{p}\left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1}\right]_0^p$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}.$$

3)

$$F(x) = \frac{x-a}{b-a}.$$

Set $F(x) = p$

$$\Rightarrow \frac{x-a}{b-a} = p$$

$$\Rightarrow x = a + p(b-a)$$

$$\Rightarrow VaR_p(X) = a + p(b-a)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p [a + t(b-a)]\,dt$$

$$= \frac{1}{p}\left[at + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= a + \frac{p}{2}(b-a).$$

4) $$F(x) = 1 - \left(\frac{K}{x}\right)^{a}$$

Set $F(x) = p$

$$\Rightarrow \quad 1 - \left(\frac{K}{x}\right)^{a} = p$$

$$\Rightarrow \quad \left(\frac{K}{x}\right)^{a} = 1 - p$$

$$\Rightarrow \quad \frac{K}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow \quad x = K(1-p)^{-\frac{1}{a}}$$

$$\Rightarrow \quad VaR_p(X) = K(1-p)^{-\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{K}{p}\int_0^p (1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{K}{p}\left[\frac{(1-t)^{1-\frac{1}{a}}}{\left(1-\frac{1}{a}\right)(-1)}\right]_0^p$$

$$= \frac{K}{p\left(\frac{1}{a}-1\right)}\left[(1-p)^{1-\frac{1}{a}} - 1\right].$$

5) The CDF of $N(0,1)$ is $\Phi(x)$.

Set $\Phi(x) = p$

$\Rightarrow \quad x = \Phi^{-1}(p)$

$\Rightarrow \quad VaR_p(X) = \Phi^{-1}(p)$.

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p \Phi^{-1}(t)\,dt$$

[Can you simplify this integral]

Set $y = \Phi^{-1}(t)$

$\Rightarrow t = \Phi(y)$

$\Rightarrow \frac{dt}{dy} = \frac{d}{dy}\Phi(y) = \phi(y)$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$$= \frac{1}{p}\int_{-\infty}^{\Phi^{-1}(p)} y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy$$

$$= \frac{1}{p\sqrt{2\pi}}\int_{-\infty}^{\Phi^{-1}(p)} y e^{-\frac{y^2}{2}}\,dy$$

$$= \frac{1}{p\sqrt{2\pi}}\left[-e^{-\frac{y^2}{2}}\right]_{-\infty}^{\Phi^{-1}(p)}$$

$$= -\frac{1}{p\sqrt{2\pi}}\, e^{-\frac{1}{2}[\Phi^{-1}(p)]^2}.$$

6)

$$F(x) = \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1}$$

Set $F(x) = p$

$$\Rightarrow \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1} = p$$

$$\Rightarrow \left(\frac{x}{a}\right)^{-b} = \frac{1}{p} - 1$$

$$\Rightarrow \frac{x}{a} = \left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$\Rightarrow VaR_p(X) = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{a}{p}\int_0^p \left(\frac{1-t}{t}\right)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p}\int_0^p t^{\frac{1}{b}}(1-t)^{-\frac{1}{b}} dt$$

Incomplete beta function

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= \frac{a}{p} B_p\left(1+\frac{1}{b}, 1-\frac{1}{b}\right).$$