MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 5

Suppose a portfolio is made up of two assets with X and Y denoting the corresponding prices. Suppose also that the joint distribution of X and Y is specified by the survival function

$$\overline{F}(x, y) = \left[1 + \frac{x}{a} + \frac{y}{b}\right]^{-c}$$

for $x > 0$, $y > 0$, $a > 0$, $b > 0$ and $c > 0$. Find the following:

1. the cdf of $M = \max(X, Y)$;
2. the pdf of M;
3. the nth moment of M;
4. the mean of M;
5. the variance of M;
6. the cdf of $L = \min(X, Y)$;
7. the pdf of L;
8. the nth moment of L;
9. the mean of L;
10. the variance of L.

$$\bar{F}(x, y) = \left[1 + \frac{x}{a} + \frac{y}{b}\right]^{-c}$$

where $x > 0$ & $y > 0$.

1. The CDF of $M = \max(X, Y)$ is

$$\begin{aligned}
F_M(m) &= P(M \leq m) \\
&= P(\max(X, Y) \leq m) \\
&= P(X \leq m, Y \leq m) \\
&= F_{X,Y}(m, m) \\
&= 1 - \bar{F}_{X,Y}(m, 0) - \bar{F}_{X,Y}(0, m) \\
&\quad + \bar{F}_{X,Y}(m, m) \\
&= 1 - \left(1 + \frac{m}{a}\right)^{-c} - \left(1 + \frac{m}{b}\right)^{-c} \\
&\quad + \left(1 + \frac{m}{a} + \frac{m}{b}\right)^{-c}.
\end{aligned}$$

2. The PDF of M is

$$\begin{aligned}
f_M(m) &= \frac{d}{dm} F_M(m) \\
&= \frac{c}{a}\left(1 + \frac{m}{a}\right)^{-c-1} + \frac{c}{b}\left(1 + \frac{m}{b}\right)^{-c-1} \\
&\quad - c\left(\frac{1}{a} + \frac{1}{b}\right)\left(1 + \frac{m}{a} + \frac{m}{b}\right)^{-c-1}
\end{aligned}$$

3) The n^{th} moment of M is

$$E(M^n) = \int_0^\infty m^n f_M(m)\, dm$$

$$= \frac{c}{a} \boxed{\int_0^\infty m^n \left(1+\frac{m}{a}\right)^{-c-1} dm}$$

$$+ \frac{c}{b} \boxed{\int_0^\infty m^n \left(1+\frac{m}{b}\right)^{-c-1} dm}$$

$$- c\left(\frac{1}{a}+\frac{1}{b}\right) \boxed{\int_0^\infty m^n \left(1+\frac{m}{a}+\frac{m}{b}\right)^{-c-1} dm}$$

Set $x = \left(1+\frac{m}{a}\right)^{-1}$

$\Rightarrow m = a \cdot \frac{1-x}{x}$

$\Rightarrow \frac{dm}{dx} = -\frac{a}{x^2}$

Set $x = \left(1+\frac{m}{b}\right)^{-1}$

$\Rightarrow m = b \cdot \frac{1-x}{x}$

$\Rightarrow \frac{dm}{dx} = -\frac{b}{x^2}$

Set $x = \left(1+\frac{m}{a}+\frac{m}{b}\right)^{-1}$

$\Rightarrow m = \frac{ab}{a+b} \frac{1-x}{x}$

$\Rightarrow \frac{dm}{dx} = -\frac{ab}{(a+b)x^2}$

$$= ca^n \int_0^1 x^{c-n-1} (1-x)^n dx$$

$$+ cb^n \int_0^1 x^{c-n-1} (1-x)^n dx$$

$$- c\left(\frac{ab}{a+b}\right)^n \int_0^1 x^{c-n-1} (1-x)^n dx$$

$$= c\left[a^n + b^n - \left(\frac{ab}{a+b}\right)^n\right] \int_0^1 x^{c-n-1} (1-x)^n dx$$

Beta function is defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

$$= c\left[a^n + b^n - \left(\frac{ab}{a+b}\right)^n\right] B(c-n, n+1).$$

4) $E(M) = c\left[a+b-\frac{ab}{a+b}\right] B(c-1, 2)$

5) $Var(M) = E(M^2) - [E(M)]^2$

$$= c\left[a^2 + b^2 - \left(\frac{ab}{a+b}\right)^2\right] B(c-2, 3)$$

$$- c^2\left[a+b-\frac{ab}{a+b}\right]^2 [B(c-1, 2)]^2.$$

6) The CDF of L is

$$F_L(\ell) = P(L \leq \ell)$$
$$= 1 - P(L > \ell)$$
$$= 1 - P(\min(X, Y) > \ell)$$
$$= 1 - P(X > \ell, Y > \ell)$$
$$= 1 - \bar{F}_{X,Y}(\ell, \ell)$$
$$= 1 - \left(1 + \frac{\ell}{a} + \frac{\ell}{b}\right)^{-c}.$$

7) The PDF of L is

$$f_L(\ell) = \frac{d}{d\ell} F_L(\ell)$$
$$= c\left(\frac{1}{a} + \frac{1}{b}\right)\left(1 + \frac{\ell}{a} + \frac{\ell}{b}\right)^{-c-1}$$

8) The nth moment of L is

$$E(L^n) = \int_0^\infty \ell^n f_L(\ell)\, d\ell$$
$$= c\left(\frac{1}{a} + \frac{1}{b}\right)\boxed{\int_0^\infty \ell^n \left(1 + \frac{\ell}{a} + \frac{\ell}{b}\right)^{-c-1} d\ell}$$

Set $x = \left(1 + \frac{\ell}{a} + \frac{\ell}{b}\right)^{-1}$

$$\Rightarrow \ell = \frac{ab}{a+b} \cdot \frac{1-x}{x}$$
$$\Rightarrow \frac{d\ell}{dx} = -\frac{ab}{(a+b)x^2}$$

$$= c\left(\frac{ab}{a+b}\right)^n \int_0^1 x^{c-n-1}(1-x)^n\,dx$$

$$= c\left(\frac{ab}{a+b}\right)^n B(c-n, n+1)$$

9) $E(L) = c \cdot \frac{ab}{a+b} B(c-1, 2)$

10) $\mathrm{Var}(L) = E(L^2) - [E(L)]^2$

$$= c\left(\frac{ab}{a+b}\right)^2 B(c-2, 3) - c^2\left(\frac{ab}{a+b}\right)^2 [B(c-1, 2)]^2 .$$