

**MATH4/68181: Extreme values and financial risk**  
**Semester 1**  
**Problem sheet for Week 4**

Suppose a portfolio contains  $\alpha$  assets valued as  $X_1, \dots, X_\alpha$ , where each is an exponential random variable with an unknown parameter  $\lambda$ . Then  $X = \max(X_1, \dots, X_\alpha)$  will be the price of the most expensive asset. Find the following:

1. the cdf of  $X$ ;
2. the pdf of  $X$ ;
3. the  $n$ th moment of  $X$ ;
4. the mean of  $X$ ;
5. the variance of  $X$ ;
6. value at risk of  $X$ ;
7. the expected shortfall of  $X$ ;
8. maximum likelihood estimates of  $\alpha$  and  $\lambda$ .

1) The CDF of  $X$  is

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(\max(X_1, \dots, X_\alpha) \leq x) \\ &= P(X_1 \leq x, \dots, X_\alpha \leq x) \\ &\stackrel{\text{indep}}{=} P(X_1 \leq x) \cdots P(X_\alpha \leq x) \\ &= (1 - e^{-\lambda x}) \cdots (1 - e^{-\lambda x}) \\ &= (1 - e^{-\lambda x})^\alpha \end{aligned}$$

2) The PDF of  $X$  is

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1} \end{aligned}$$

3) The  $n$ th moment of  $X$  is

$$\begin{aligned} E(X^n) &= \int_0^\infty x^n f_X(x) dx \\ &= \alpha \lambda \int_0^\infty x^n e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1} dx \end{aligned}$$

$\text{Set } y = e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda} \log y \Rightarrow \frac{dx}{dy} = -\frac{1}{\lambda y}$

$$= \cancel{\lambda} \int_1^0 \left(-\frac{1}{\lambda} \log y\right)^n y (1-y)^{\alpha-1} \left(-\frac{1}{\lambda y}\right) dy$$

$$= \frac{\alpha(-1)^n}{\lambda^n} \int_0^1 (\log y)^n (1-y)^{\alpha-1} dy$$

$$\boxed{(\log y)^n = \frac{\partial^n}{\partial \beta^n} y^\beta \Big|_{\beta=0}}$$

$$= \frac{\alpha(-1)^n}{\lambda^n} \int_0^1 \left( \frac{\partial^n}{\partial \beta^n} y^\beta \Big|_{\beta=0} \right) \cdot (1-y)^{\alpha-1} dy$$

$$= \frac{\alpha(-1)^n}{\lambda^n} \frac{\partial^n}{\partial \beta^n} \left[ \int_0^1 y^\beta (1-y)^{\alpha-1} dy \right] \Big|_{\beta=0}$$

$$\boxed{B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt}$$

$$= \frac{\alpha(-1)^n}{\lambda^n} \frac{\partial^n}{\partial \beta^n} B(\beta+1, \alpha) \Big|_{\beta=0}$$

4) Mean of  $X = E(X)$

$$= -\frac{\alpha}{\lambda} \frac{\partial}{\partial \beta} B(\beta+1, \alpha) \Big|_{\beta=0}$$

$$= -\frac{\alpha}{\lambda} \frac{\partial}{\partial \beta} \left[ \frac{\Gamma(\beta+1) \Gamma(\alpha)}{\Gamma(\beta+1+\alpha)} \right] \Big|_{\beta=0}$$

$$= -\frac{\alpha}{\lambda} \left[ \frac{\Gamma'(1)}{\alpha} - \frac{\Gamma(\alpha) \Gamma'(\alpha+1)}{(\Gamma(\alpha+1))^2} \right]$$

$$5) \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= + \frac{\alpha}{\lambda^2} \frac{\partial^2}{\partial \beta^2} B(\beta+1, \alpha) \Big|_{\beta=0} - [E(X)]^2$$

$$= + \frac{\alpha}{\lambda^2} \frac{\partial^2}{\partial \beta^2} \left[ \frac{\Gamma(\beta+1) \Gamma(\alpha)}{\Gamma(\beta+1+\alpha)} \right] \Big|_{\beta=0} - [E(X)]^2$$

$$= + \frac{\alpha}{\lambda^2} \left[ \frac{\Gamma''(1)}{\alpha} - \frac{2 \Gamma(\alpha) \Gamma'(1) \Gamma'(\alpha+1)}{(\Gamma(\alpha+1))^2} \right. \\ \left. - \frac{\Gamma(\alpha) \Gamma''(\alpha+1)}{(\Gamma(\alpha+1))^2} \right. \\ \left. + \frac{2 \Gamma(\alpha) (\Gamma'(\alpha+1))^2}{(\Gamma(\alpha+1))^3} \right] - [E(X)]^2$$

6) Not yet

7) " "



8) Supposing  $x_1, \dots, x_n$  is a random sample on  $X$  (the most expensive asset), find the MLEs of  $\alpha$  and  $\lambda$ .

The likelihood function is

$$L(\alpha, \lambda) = \prod_{i=1}^n [\alpha \lambda e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{\alpha-1}]$$

$$= \alpha^n \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \left[ \prod_{i=1}^n (1 - e^{-\lambda x_i}) \right]^{\alpha-1}$$

The log-likelihood function is

$$\log L(\alpha, \lambda) = n \log \alpha + n \log \lambda - \lambda \sum_{i=1}^n x_i + (\alpha - 1) \sum_{i=1}^n \log(1 - e^{-\lambda x_i})$$

The partial derivatives wrt  $\alpha$  and  $\lambda$  are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log(1 - e^{-\lambda x_i}) = 0 \quad (1)$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i + (\alpha - 1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} = 0 \quad (2)$$

$$(1) \Rightarrow \hat{\alpha} = - \frac{n}{\sum_{i=1}^n \log(1 - e^{-\hat{\lambda} x_i})} \quad (3)$$

Substitute (3) into (2)  $\Rightarrow$

$$\frac{n}{\hat{\alpha}} - \sum_{i=1}^n x_i - \left( \frac{n}{\sum_{i=1}^n \log(1 - e^{-\hat{\lambda} x_i})} + 1 \right) \sum_{i=1}^n \frac{x_i e^{-\hat{\lambda} x_i}}{1 - e^{-\hat{\lambda} x_i}} = 0 \quad (4)$$

Solve (4) to get  $\hat{\lambda}$ . Use (3) to obtain  $\hat{\alpha}$ .