MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 3

1. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Bernoulli (p) and let $M_n = \max(X_1, X_2, \ldots, X_n)$. Determine if $(M_n - b_n)/a_n$ for suitable norming constants a_n and b_n will have a non-degenerate limiting distribution as $n \to \infty$.

2. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the degenerate distribution specified by the pmf

$$p(k) = \begin{cases} 1, & \text{if } k = k_0, \\ 0, & \text{if } k \neq k_0. \end{cases}$$

Determine if $(M_n - b_n)/a_n$ for suitable norming constants a_n and b_n will have a non-degenerate limiting distribution as $n \to \infty$.

3. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Yule distribution specified by the pmf

$$p(k) = \rho B(k, \rho + 1), \ k \geq 1.$$

Determine if $(M_n - b_n)/a_n$ for suitable norming constants a_n and b_n will have a non-degenerate limiting distribution as $n \to \infty$.

4. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the zeta distribution specified by the pmf

$$p(k) = k^{-s}/\zeta(s), \ k \geq 1,$$

where $\zeta(\cdot)$ denotes the Riemann zeta function. Determine if $(M_n - b_n)/a_n$ for suitable norming constants a_n and b_n will have a non-degenerate limiting distribution as $n \to \infty$.

5. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Gauss-Kuzmin distribution specified by the pmf

$$p(k) = -\log_2 \left[1 - (k+1)^{-2}\right], \ k \geq 1,$$

and the cdf

$$F(k) = 1 - \log_2 \left[\frac{k+2}{k+1}\right], \ k \geq 1.$$

Determine if $(M_n - b_n)/a_n$ for suitable norming constants a_n and b_n will have a non-degenerate limiting distribution as $n \to \infty$.

6. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the discrete Lindley distribution specified by the pmf

$$p(x) = \frac{p^x}{1+\theta} \left\{\theta(1-2p) + (1-p)(1+\theta x)\right\},$$

where $p = \exp(-\theta)$, for $\theta > 0$ and $x = 0, 1, 2, \ldots$. Determine if $(M_n - b_n)/a_n$ for suitable norming constants a_n and b_n will have a non-degenerate limiting distribution as $n \to \infty$.

7. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the discrete Weibull distribution specified by the cdf

$$F(x) = 1 - q^{(x+1)^a},$$

where $0 < q < 1$, for $a > 0$ and $x = 0, 1, 2, \ldots$. Determine if $(M_n - b_n)/a_n$ for suitable norming constants a_n and b_n will have a non-degenerate limiting distribution as $n \to \infty$.

1) $X_1, \ldots, X_n$ IID Bernoulli (p).

Does the ETT hold?

$$\boxed{\lim_{k \to w(F)} \frac{P(X = k)}{1 - F(k-1)} \neq 0 \Rightarrow \text{ETT fails to hold}}$$

$X \sim$ Bernoulli(p) if

$$P(X = k) = \begin{cases} 1-p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}$$

$$\Rightarrow \quad w(F) = 1$$

$$\Rightarrow \quad \lim_{k \to 1} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \frac{P(X = 1)}{1 - F(0)}$$

$$= \frac{p}{1 - P(X \leq 0)}$$

$$= \frac{p}{1 - P(X = 0)}$$

$$= \frac{p}{1 - (1-p)} = 1 \neq 0$$

$\Rightarrow$ Hence, the ETT fails to hold.

2) $X_1, \ldots, X_n$ IID with PMF

$$P(X=k) = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq k_0 \end{cases}$$

Does the ETT hold?

$$w(F) = k_0$$

$$\Rightarrow \lim_{k \to k_0} \frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{P(X=k_0)}{1-F(k_0-1)}$$

$$= \frac{1}{1-P(X \leq k_0 - 1)}$$

$$= \frac{1}{1-0} \neq 0$$

$\Rightarrow$ Hence, the ETT fails to hold.

3) Too hard for your level.

Ignore it.

4) $X_1, \ldots, X_n$ IID with PMF

$$P(X=k) = \frac{k^{-s}}{\zeta(s)}, \quad k = 1, 2, \ldots$$

where $\zeta(\cdot)$ denotes the zeta function.

$$\omega(F) = +\infty.$$

$$\Rightarrow \lim_{k \to \infty} \frac{P(X=k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{1 - P(X \leq k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{P(X \geq k)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{\sum_{i=k}^{\infty} P(X=i)}$$

$$= \lim_{k \to \infty} \frac{\frac{k^{-s}}{\cancel{\zeta(s)}}}{\sum_{i=k}^{\infty} \frac{i^{-s}}{\cancel{\zeta(s)}}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\sum_{i=k}^{\infty} i^{-s}}$$

$$\boxed{\begin{array}{l}\displaystyle\sum_{i=k}^{\infty} g(i) \approx \int_k^{\infty} g(x)\,dx \\ \text{as } k \to \infty \text{ under certain conditions}\end{array}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\int_k^{\infty} x^{-s}\,dx}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{\left[\frac{x^{1-s}}{1-s}\right]_k^{\infty}}$$

$$= \lim_{k \to \infty} \frac{k^{-s}}{0 - \frac{k^{1-s}}{1-s}} \qquad \text{provided } \underline{1-s < 0}$$

$$= \lim_{k \to \infty} \frac{s-1}{k}$$

$$= 0$$

Hence the ETT does hold.

Check conditions I-$\underline{III}$ to see which limit will be attained.

5) Suppose $X_1, \dots, X_n$ IID with PMF and CDF

$$p(k) = -\log_2\left[1 - (k+1)^{-2}\right]$$

and

$$CDF = 1 - \log_2\left(\frac{k+2}{k+1}\right),$$

for $k = 1, 2, \dots$ Does the ETT hold?

$$w(F) = +\infty$$

$$\Rightarrow \lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)}$$

$$= \lim_{k\to\infty} \frac{-\log_2\left[1-(k+1)^{-2}\right]}{\cancel{1} - \left[\cancel{1} - \log_2\left(\frac{k+1}{k}\right)\right]}$$

$$= \lim_{k\to\infty} \frac{-\log_2\left[\frac{(k+1)^2-1}{(k+1)^2}\right]}{\log_2\left(\frac{k+1}{k}\right)}$$

$$= \lim_{k\to\infty} \frac{-\log_2\left[\frac{k^2+2k}{(k+1)^2}\right]}{\log_2\left(\frac{k+1}{k}\right)}$$

$$= \lim_{k\to\infty} \frac{-\log_2(k^2+2k) + 2\log_2(k+1)}{\log_2(k+1) - \log_2 k}$$

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{(\log_e a)}\cdot\frac{1}{x}}$$

$$\underset{\text{rule}}{\overset{\text{L'H}}{=}} \lim_{k\to\infty} \frac{-\frac{2k+2}{(\cancel{\log 2})(k^2+2k)} + \frac{2}{(\cancel{\log 2})(k+1)}}{\frac{1}{(\cancel{\log 2})(k+1)} - \frac{1}{(\cancel{\log 2})k}}$$

$$= \lim_{k\to\infty} \frac{-\frac{2(k+1)}{k(k+2)} + \frac{2}{k+1}}{\frac{1}{k+1} - \frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{-\frac{2(k+1)}{k(k+2)} + \frac{2}{k+1}}{-\frac{1}{k(k+1)}}$$

$$= \lim_{k\to\infty} \left[\frac{2(k+1)^2}{k+2} - 2k\right]$$

$$= \lim_{k\to\infty} 2\cdot\frac{[(k+1)^2 - k(k+2)]}{k+2}$$

$$= \lim_{k\to\infty} 2\cdot\frac{[\cancel{k^2}+\cancel{2k}+1-\cancel{k^2}-\cancel{2k}]}{k+2}$$

$$= \lim_{k\to\infty} \frac{2}{k+2} = 0$$

Hence, the ETT holds.
Check conditions I–III to see which limit will be attained.

6) A bit too hard.

Please ignore.

7) Suppose $X_1, \dots, X_n$ IID with CDF

$$F(x) = 1 - q^{(x+1)^a}$$

where $0 < q < 1$, $a > 0$ and $x = 0, 1, 2, \dots$

$$w(F) = +\infty$$

$$\Rightarrow \lim_{k \to \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$\boxed{\begin{array}{l}\text{Suppose } X \text{ is a discrete RV with positive integers as support. Then} \\ P(X = k) = F(k) - F(k-1)\end{array}}$$

$$= \lim_{k \to \infty} \frac{F(k) - F(k-1)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{\cancel{1} - q^{(k+1)^a} - [\cancel{1} - q^{k^a}]}{\cancel{1} - [\cancel{1} - q^{k^a}]}$$

$$= \lim_{k \to \infty} \frac{q^{k^a} - q^{(k+1)^a}}{q^{k^a}}$$

$$= \lim_{k \to \infty} \left[1 - q^{(k+1)^a - k^a} \right]$$

$$= 1 - \lim_{k \to \infty} q^{k^a\left(1+\frac{1}{k}\right)^a - k^a}$$

$$= 1 - \lim_{k \to \infty} q^{k^a\left[\left(1+\frac{1}{k}\right)^a - 1\right]}$$

$$\boxed{\begin{array}{l}\text{Binomial series} \\ (1+x)^a = \sum_{i=0}^{\infty} \binom{a}{i} x^i\end{array}}$$

$$= 1 - \lim_{k \to \infty} q^{k^a\left[\sum_{i=0}^{\infty}\binom{a}{i}\left(\frac{1}{k}\right)^i - 1\right]}$$

$$= 1 - \lim_{k \to \infty} q^{k^a\left[\cancel{1} + \frac{a}{k} + \frac{a(a-1)}{2k^2} + \cdots - \cancel{1}\right]}$$

$$= 1 - \lim_{k \to \infty} q^{k^a \cdot \frac{a}{k}}$$

$$= 1 - \lim_{k \to \infty} q^{a k^{a-1}}$$

$$= \begin{cases} 1 - q^{a \cdot \infty} = 1 - 0 = 1 & a > 1 \\ 1 - q & a = 1 \\ 1 - q^0 = 1 - 1 = 0 & a < 1 \end{cases}$$

Hence, the ETT fails if $a \geq 1$

the ETT holds if $a < 1$

If $a < 1$ check conditions I - III to see which limit will be attained.