MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 2

1. Find the density functions of $\Lambda(x)$, $\Phi_\alpha(x)$, and $\Psi_\alpha(x)$.
2. Find the means corresponding to $\Lambda(x)$, $\Phi_\alpha(x)$, and $\Psi_\alpha(x)$.
3. Find the variances corresponding to $\Lambda(x)$, $\Phi_\alpha(x)$, and $\Psi_\alpha(x)$.
4. Show that $\Lambda^n(x) = \Lambda\left(\alpha_n x + \beta_n\right)$ if and only if $\alpha_n = 1$ and $\beta_n = -\log n$.
5. Show that $\Phi_\alpha^n(x) = \Phi_\alpha\left(\alpha_n x + \beta_n\right)$ if and only if $\alpha_n = n^{-1/\alpha}$ and $\beta_n = 0$.
6. Show that $\Psi_\alpha^n(x) = \Psi_\alpha\left(\alpha_n x + \beta_n\right)$ if and only if $\alpha_n = n^{1/\alpha}$ and $\beta_n = 0$.
7. Find the max domain of attraction of the exponential cdf $F(x) = 1 - \exp(-x)$.
8. Find the max domain of attraction of the exponentiated exponential cdf $F(x) = [1-\exp(-x)]^\alpha$.
9. Find the max domain of attraction of the uniform$[0,1]$ cdf $F(x) = x$.
10. Find the max domain of attraction of the Pareto cdf $F(x) = 1 - (K/x)^\alpha$.
11. Consider a class of distributions defined by the cdf

$$F(x) = K \int_0^{G(x)} t^{a-1}(1-t)^{b-1} \exp(-ct)dt,$$

and the pdf

$$f(x) = Kg(x)G(x)^{a-1}\left\{1 - G(x)\right\}^{b-1} \exp\left\{-c\, G(x)\right\},$$

where $a > 0$, $b > 0$, $-\infty < c < \infty$, $G(\cdot)$ is a valid cdf and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G. You may assume $w(F) = w(G)$.

12. Consider a class of distributions defined by the cdf

$$F(x) = \frac{\beta^a}{B(a,b)} \int_{-\infty}^{x} \frac{g(t)\left[G(t)\right]^{a-1}\left[1 - G(t)\right]^{b-1}}{\left[1-(1-\beta)G(t)\right]^{a+b}} dt$$

and the pdf

$$f(x) = \frac{\beta^a}{B(a,b)} \frac{g(x)\left[G(x)\right]^{a-1}\left[1 - G(x)\right]^{b-1}}{\left[1-(1-\beta)G(x)\right]^{a+b}}.$$

where $a > 0$, $b > 0$, $\beta > 0$, $G(\cdot)$ is a valid cdf and $g(x) = dG(x)/dx$. Show that F belongs to the same max domain of attraction as G. You may assume $w(F) = w(G)$.

13. If

$$G(x) = \exp\left\{-\left(1 + \xi\frac{x-\mu}{\sigma}\right)^{-1/\xi}\right\},$$

the GEV cdf, show that

$$G^{-1}(p) = \mu - \frac{\sigma}{\xi}\left[1 - \left\{-\log p\right\}^{-\xi}\right].$$

14. If

$$G(x) = 1 - \left\{1 + \xi \frac{x - t}{\sigma}\right\}^{-1/\xi},$$

the GP cdf, show that

$$G^{-1}(p) = t + \frac{\sigma}{\xi}\left\{(1-p)^{-\xi} - 1\right\}.$$

1) Find the PDFs of $\Lambda(x)$, $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$.

$$\Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

$$\frac{d\Lambda(x)}{dx} = e^{-x} e^{-e^{-x}}$$

$$\Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\frac{d}{dx}\Phi_\alpha(x) = \alpha x^{-\alpha-1} e^{-x^{-\alpha}}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

$$\frac{d}{dx}\Psi_\alpha(x) = \alpha(-x)^{\alpha-1} e^{-(-x)^\alpha}$$

2) Find the means corresponding to $\Lambda(x)$, $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$.

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot e^{-x} e^{-e^{-x}} \, dx$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= e^{-x} \\ x &= -\log y \\ \frac{dx}{dy} &= -\frac{1}{y} \end{aligned}}$$

$$= \int_{\infty}^{0} (-\log y) \cancel{y} e^{-y} \left(-\frac{1}{\cancel{y}}\right) dy$$

$$= -\int_{0}^{\infty} (\log y) \, e^{-y} \, dy$$

$$\boxed{\left.\frac{d}{da} y^a \right|_{a=0} = \log y}$$

$$= -\int_{0}^{\infty} \left.\frac{d}{da} y^a \right|_{a=0} e^{-y} dy$$

$$= -\frac{d}{da}\left[\int_{0}^{\infty} y^a e^{-y} dy\right]\Bigg|_{a=0}$$

$$= -\frac{d}{da} \Gamma(a+1)\Big|_{a=0}$$

$$= -\Gamma'(1)$$

$$\text{Mean} = \int_0^\infty x \cdot \alpha x^{-\alpha-1} e^{-x^{-\alpha}} \, dx$$

$$= \alpha \int_0^\infty x^{-\alpha} e^{-x^{-\alpha}} \, dx$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= x^{-\alpha} \\ x &= y^{-\frac{1}{\alpha}} \\ \frac{dx}{dy} &= -\frac{1}{\alpha} y^{-\frac{1}{\alpha}-1} \end{aligned}}$$

$$= \cancel{\alpha} \int_\infty^0 y e^{-y} \left(-\cancel{\tfrac{1}{\alpha}} y^{-\frac{1}{\alpha}-1} \right) dy$$

$$= \int_0^\infty y^{-\frac{1}{\alpha}} e^{-y} \, dy$$

$$= \Gamma\left(1 - \frac{1}{\alpha}\right).$$

$$\text{Mean} = \int_{-\infty}^{0} x \cdot \alpha (-x)^{\alpha-1} e^{-(-x)^{\alpha}} \, dx$$

$$= \alpha \int_{-\infty}^{0} x \cdot (-x)^{\alpha-1} e^{-(-x)^{\alpha}} \, dx$$

$$\boxed{\begin{aligned} &\text{Set } y = (-x)^{\alpha} \\ &\Rightarrow x = -y^{\frac{1}{\alpha}} \\ &\Rightarrow \frac{dx}{dy} = -\frac{1}{\alpha} y^{\frac{1}{\alpha}-1} \end{aligned}}$$

$$= \alpha \int_{\infty}^{0} \left(-y^{\frac{1}{\alpha}}\right) y^{\frac{\alpha-1}{\alpha}} e^{-y} \left(-\frac{1}{\alpha} y^{\frac{1}{\alpha}-1}\right) dy$$

$$= -\int_{0}^{\infty} y^{\frac{1}{\alpha}} e^{-y} \, dy$$

$$= -\Gamma\left(1+\frac{1}{\alpha}\right).$$

3) Find variances corresponding to $\Lambda(x)$, $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$.

$$\text{Variance} = \int_{-\infty}^{\infty} x^2 \cdot e^{-x} e^{-e^{-x}} dx - [\Gamma'(1)]^2$$

$$\boxed{\begin{aligned} &\text{Set } y = e^{-x} \\ &\Rightarrow x = -\log y \\ &\Rightarrow \frac{dx}{dy} = -\frac{1}{y} \end{aligned}}$$

$$= \int_{\infty}^{0} (\log y)^2 \cdot y\, e^{-y} \left(-\frac{1}{y}\right) dy - [\Gamma'(1)]^2$$

$$= \int_0^{\infty} (\log y)^2 e^{-y}\, dy - [\Gamma'(1)]^2$$

$$\boxed{\frac{d^2}{da^2} y^a \Big|_{a=0} = (\log y)^2}$$

$$= \int_0^{\infty} \frac{d^2}{da^2} y^a \Big|_{a=0} e^{-y}\, dy - [\Gamma'(1)]^2$$

$$= \frac{d^2}{da^2} \left[\int_0^{\infty} y^a e^{-y}\, dy\right] \Big|_{a=0} - [\Gamma'(1)]^2$$

$$= \frac{d^2}{da^2} \Gamma(a+1) \Big|_{a=0} - [\Gamma'(1)]^2$$

$$= \Gamma''(1) - [\Gamma'(1)]^2.$$

$$\text{Variance} = \int_0^\infty x^2 \cdot \alpha x^{-\alpha-1} e^{-x^{-\alpha}} dx - \left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^2$$

$$= \alpha \int_0^\infty x^{1-\alpha} e^{-x^{-\alpha}} dx - \left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^2$$

$$\boxed{\begin{aligned} &\text{Set } y = x^{-\alpha} \\ &\Rightarrow x = y^{-\frac{1}{\alpha}} \\ &\Rightarrow \frac{dx}{dy} = -\frac{1}{\alpha} y^{-\frac{1}{\alpha}-1} \end{aligned}}$$

$$= \alpha \int_\infty^0 y^{\frac{\alpha-1}{\alpha}} e^{-y} \left(-\frac{1}{\alpha} y^{-\frac{1}{\alpha}-1}\right) dy - \left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^2$$

$$= \int_0^\infty y^{-\frac{2}{\alpha}} e^{-y} dy - \left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^2$$

$$= \Gamma\left(1-\frac{2}{\alpha}\right) - \left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^2 .$$

$$\text{Variance} = \int_{-\infty}^{0} x^2 \cdot \alpha (-x)^{\alpha-1} e^{-(-x)^{\alpha}} \, dx - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2$$

$$= \alpha \int_{-\infty}^{0} (-x)^{\alpha+1} e^{-(-x)^{\alpha}} \, dx - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2$$

$$\boxed{\begin{aligned} &\text{Set} \quad y = (-x)^{\alpha} \\ &\Rightarrow \quad x = -y^{\frac{1}{\alpha}} \\ &\Rightarrow \quad \frac{dx}{dy} = -\frac{1}{\alpha} y^{\frac{1}{\alpha}-1} \end{aligned}}$$

$$= \alpha \int_{\infty}^{0} y^{\frac{\alpha+1}{\alpha}} e^{-y} \left(-\frac{1}{\alpha} y^{\frac{1}{\alpha}-1}\right) dy - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2$$

$$= \int_{0}^{\infty} y^{\frac{2}{\alpha}} e^{-y} \, dy - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2$$

$$= \Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2.$$

4) $\Lambda^n(x) = \Lambda(\alpha_n x + \beta_n)$

$$\Leftrightarrow \left[e^{-e^{-x}}\right]^n = e^{-e^{-(\alpha_n x + \beta_n)}}$$

$$\Leftrightarrow e^{-ne^{-x}} = e^{-e^{-\alpha_n x - \beta_n}}$$

$$\Leftrightarrow -ne^{-x} = -e^{-\alpha_n x - \beta_n}$$

$$\Leftrightarrow ne^{-x} = e^{-\alpha_n x - \beta_n}$$

$$\Leftrightarrow \log n - x = -\alpha_n x - \beta_n$$

$$\Leftrightarrow \log n = -\beta_n \ \& \ -1 = -\alpha_n$$

$$\Leftrightarrow \beta_n = -\log n \ \& \ \alpha_n = 1$$

5) $\left[\Phi_\alpha(x)\right]^n = \Phi_\alpha(\alpha_n x + \beta_n)$

$$\Leftrightarrow \left[e^{-x^{-\alpha}}\right]^n = e^{-(\alpha_n x + \beta_n)^{-\alpha}}$$

$$\Leftrightarrow e^{-nx^{-\alpha}} = e^{-(\alpha_n x + \beta_n)^{-\alpha}}$$

$$\Leftrightarrow -nx^{-\alpha} = -(\alpha_n x + \beta_n)^{-\alpha}$$

$$\Leftrightarrow nx^{-\alpha} = (\alpha_n x + \beta_n)^{-\alpha}$$

$$\Leftrightarrow \left[nx^{-\alpha}\right]^{-\frac{1}{\alpha}} = \left[(\alpha_n x + \beta_n)^{-\alpha}\right]^{-\frac{1}{\alpha}}$$

$$\Leftrightarrow n^{-\frac{1}{\alpha}} x = \alpha_n x + \beta_n$$

$$\Leftrightarrow 0 = \beta_n \quad \& \quad n^{-\frac{1}{\alpha}} = \alpha_n$$

$$\Leftrightarrow \beta_n = 0 \quad \& \quad \alpha_n = n^{-\frac{1}{\alpha}}.$$

6) $[\Psi_\alpha(x)]^n = \Psi_\alpha(\alpha_n x + \beta_n)$

$$\Leftrightarrow \left[e^{-(-x)^\alpha}\right]^n = e^{-(-(\alpha_n x+\beta_n))^\alpha}$$

$$\Leftrightarrow e^{-n(-x)^\alpha} = e^{-(-\alpha_n x - \beta_n)^\alpha}$$

$$\Leftrightarrow -n(-x)^\alpha = -(-\alpha_n x - \beta_n)^\alpha$$

$$\Leftrightarrow n(-x)^\alpha = (-\alpha_n x - \beta_n)^\alpha$$

$$\Leftrightarrow [n(-x)^\alpha]^{\frac{1}{\alpha}} = [(-\alpha_n x - \beta_n)^\alpha]^{\frac{1}{\alpha}}$$

$$\Leftrightarrow n^{\frac{1}{\alpha}}(-x) = -\alpha_n x - \beta_n$$

$$\Leftrightarrow n^{\frac{1}{\alpha}} x = \alpha_n x + \beta_n$$

$$\Leftrightarrow 0 = \beta_n \;\&\; n^{\frac{1}{\alpha}} = \alpha_n$$

$$\Leftrightarrow \beta_n = 0 \;\&\; \alpha_n = n^{\frac{1}{\alpha}}.$$

7) Find the limit for $F(x) = 1 - e^{-x}$, $x > 0$;
$\omega(F) = +\infty$.

$\underline{I}$:
$$\lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - [\cancel{1} - e^{-t - x\gamma(t)}]}{\cancel{1} - [\cancel{1} - e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1$$

Hence, condition I holds. That is, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n \to \infty} [F(a_n x + b_n)]^n = e^{-e^{-x}}$$

8) Find the limit for $F(x) = [1-e^{-x}]^{\alpha}$, $x > 0$

$$w(F) = +\infty$$

$$I: \lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - [1 - \boxed{e^{-t - x\gamma(t)}}]^{\alpha}}{1 - [1 - \boxed{e^{-t}}]^{\alpha}} \quad \to 0$$

$$\boxed{(1-a)^{\alpha} \sim 1 - \alpha \cdot a \quad \text{if} \quad a \to 0}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - [\cancel{1} - \cancel{\alpha} \cdot e^{-t - x\gamma(t)}]}{\cancel{1} - [\cancel{1} - \cancel{\alpha} \cdot e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1 \quad \forall t.$$

Hence, condition I holds. That is, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n \to \infty} [F(a_n x + b_n)]^n = e^{-e^{-x}}.$$

9) Find the limit for $F(x) = x,\ 0 < x < 1$

$w(F) = 1$

I: $\lim_{t \to 1} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$= \lim_{t \to 1} \frac{1 - (t + x\gamma(t))}{1 - t}$

$= \lim_{t \to 1} \left[1 - \frac{x\gamma(t)}{1-t}\right]$

$\neq e^{-x}$

$\Rightarrow$ condition I fails to hold.

II: $\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$

$= \lim_{t \to \infty} \frac{1 - tx}{1 - t}$

$= \lim_{t \to \infty} \frac{\frac{1}{t} - x}{\frac{1}{t} - 1}$

$\frac{1}{t} \to 0$

$= x \neq x^{-\alpha}$

$\Rightarrow$ condition II fails to hold.

Plus $w(F) = 1 \neq \infty$

$\underline{\text{III}}$: $\omega(F) = 1 < \infty$ ✓

$$\lim_{t \to 0} \frac{1 - F(1 - tx)}{1 - F(1 - t)}$$

$$= \lim_{t \to 0} \frac{\cancel{1} - (\cancel{1} - tx)}{\cancel{1} - (\cancel{1} - t)}$$

$$= \lim_{t \to 0} \frac{tx}{t}$$

$$= x = x^{\alpha} \quad \text{if} \quad \alpha = 1$$

Hence, condition $\underline{\text{III}}$ holds. That is, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n \to \infty} \left[F(a_n x + b_n) \right]^n \longrightarrow \begin{cases} e^{x} & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

10) Find the limit for $F(x) = 1 - \left(\frac{K}{x}\right)^{\alpha}, x \geq K$

$\omega(F) = +\infty$

$$I: \lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{t + x\gamma(t)}\right)^{\alpha}\right]}{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{t}\right)^{\alpha}\right]}$$

$$= \lim_{t \to \infty} \left(\frac{t}{t + x\gamma(t)}\right)^{\alpha}$$

$\neq e^{-x}$

$\Rightarrow$ condition $\underline{I}$ fails to hold.

II : $\quad \omega(F) = +\infty \ \checkmark$

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{tx}\right)^{\alpha}\right]}{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{t}\right)^{\alpha}\right]}$$

$$= \lim_{t \to \infty} \left(\frac{t}{tx}\right)^{\alpha}$$

$$= x^{-\alpha}$$

Hence, condition II holds. That is, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n \to \infty} \left[F(a_n x + b_n)\right]^n \longrightarrow \begin{cases} e^{-x^{-\alpha}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

11). Consider the class of distributions with CDF and PDF given by

$$F(x) = K \int_0^{G(x)} t^{a-1} (1-t)^{b-1} e^{-ct} \, dt$$

and

$$f(x) = K \, g(x) \left[G(x) \right]^{a-1} \left[1 - G(x) \right]^{b-1} e^{-c G(x)}$$

where $a > 0$, $b > 0$, $c > 0$, G is a valid CDF and $g(x) = \frac{d\,G(x)}{dx}$. Show that F and G belong to the same domain of attraction. We assume $\omega(F) = \omega(G)$.

(A) Assume G belongs to Gumbel limit and show F also " " " "

(B) Assume G belongs to Frēchet limit and show F also " " " "

(C) Assume G belongs to Weibull limit and show F also " " " "

A) Assume G belongs to Gumbel limit. That is, there exist $\gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad \cdots (*)$$

We have

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\underset{\text{rule}}{\overset{L'H}{=}} \lim_{t \to w(F)} \frac{-f(t + x\gamma(t))(1 + x\gamma'(t))}{-f(t)}$$

$$= \lim_{t \to w(F)} \frac{f(t + x\gamma(t))}{f(t)} \cdot (1 + x\gamma'(t))$$

$$= \lim_{t \to w(F)} \frac{\cancel{K} g(t + x\gamma(t)) [G(t + x\gamma(t))]^{a-1} [1 - G(t + x\gamma(t))]^{b-1}}{\cancel{K} g(t) [G(t)]^{a-1} [1 - G(t)]^{b-1}}$$

$$\cdot \frac{e^{-cG(t + x\gamma(t))}}{e^{-cG(t)}} \cdot (1 + x\gamma'(t))$$

$$\overset{w(F)}{\underset{= w(G)}{=}} \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} \left[\frac{\boxed{G(t + x\gamma(t))} \to 1}{\boxed{G(t)} \to 1} \right]^{a-1}$$

$$\cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{b-1} e^{c\boxed{G(t)} - c\boxed{G(t + x\gamma(t))}} \cdot (1 + x\gamma'(t))$$

$\boxed{G(t)} \to 1$, $\boxed{G(t + x\gamma(t))} \to 1$

$$= \lim_{t \to w(G)} \left[\frac{g(t + x\gamma(t))}{g(t)} \left(1 + x\gamma'(t)\right) \right] \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{b-1}$$

$$\overset{\text{L'H rule in reverse}}{=} \lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \cdot \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{b-1}$$

$$= \lim_{t \to w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)} \right]^{b}$$

$$\overset{(*)}{=} \left[e^{-x} \right]^{b}$$

$$= e^{-bx}$$

$=$ same type as e^{-x}

Hence, F also belongs to Gumbel limit.

B) Assume G belongs to Frechet limit. That is,

$$\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\alpha}, \quad \alpha > 0 \qquad \ldots \qquad (**)$$

We have

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$\overset{L'H}{\underset{\text{rule}}{=}} \lim_{t \to \infty} \frac{-x f(tx)}{-f(t)}$$

$$= \lim_{t \to \infty} \frac{x f(tx)}{f(t)}$$

$$= \lim_{t \to \infty} \frac{x \cancel{K} g(tx) [G(tx)]^{a-1} [1 - G(tx)]^{b-1} e^{-cG(tx)}}{\cancel{K} g(t) [G(t)]^{a-1} [1 - G(t)]^{b-1} e^{-cG(t)}}$$

$$= \lim_{t \to \infty} x \frac{g(tx)}{g(t)} \left[\frac{\overset{\to 1}{\boxed{G(tx)}}}{\underset{\to 1}{\boxed{G(t)}}} \right]^{a-1} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{b-1} e^{c \overset{\to 1}{\boxed{G(t)}} - c \overset{\to 1}{\boxed{G(tx)}}}$$

$$= \lim_{t \to \infty} \left[\frac{x g(tx)}{g(t)} \right] \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{b-1}$$

$$\overset{L'H}{\underset{\text{rule in reverse}}{=}} \lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{b-1}$$

$$= \lim_{t \to \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^{b}$$

$\stackrel{(**)}{=}$ $\left[x^{-\alpha}\right]^{b}$

$= x^{-b\alpha}$

$=$ same type as $x^{-\alpha}$

Hence, F also belongs to Frechet limit.

c) Assume G belongs to Weibull limit. That is,

$$\lim_{t\to 0} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)} = x^{\alpha}, \quad \alpha>0 \qquad \cdots \quad (***)$$

We have

$$\lim_{t\to 0} \frac{1-F(w(F)-tx)}{1-F(w(F)-t)}$$

$$\overset{w(F)=w(G)}{=} \lim_{t\to 0} \frac{1-F(w(G)-tx)}{1-F(w(G)-t)}$$

$$\overset{\text{L'H rule}}{=} \lim_{t\to 0} \frac{x\, f(w(G)-tx)}{f(w(G)-t)}$$

$$= \lim_{t\to 0} \frac{x\,\cancel{K}\, g(w(G)-tx)\,[G(w(G)-tx)]^{a-1}[1-G(w(G)-tx)]^{b-1}}{\cancel{K}\, g(w(G)-t)\,[G(w(G)-t)]^{a-1}[1-G(w(G)-t)]^{b-1}} \cdot \frac{e^{-cG(w(G)-tx)}}{e^{-cG(w(G)-t)}}$$

$$= \lim_{t\to 0} x\,\frac{g(w(G)-tx)}{g(w(G)-t)} \left[\underbrace{\frac{\boxed{G(w(G)-tx)}}{\boxed{G(w(G)-t)}}}_{\to 1}\right]^{a-1} \left[\frac{1-G(w(G)-tx)}{1-G(w(G)-t)}\right]^{b-1} e^{c\,\underbrace{\boxed{G(w(G)-t)}}_{\downarrow 1} - c\,\underbrace{\boxed{G(w(G)-tx)}}_{\downarrow 1}}$$

$$= \lim_{t \to 0} \left[x \, \frac{g(w(G) - tx)}{g(w(G) - t)} \right] \left[\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \right]^{b-1}$$

$$\underset{\substack{\text{rule} \\ \text{in} \\ \text{reverse}}}{\overset{L'H}{=}} \lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \left[\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \right]^{b-1}$$

$$= \lim_{t \to 0} \left[\frac{1 - G(w(G) - tx)}{1 - G(w(G)} \right]^{b}$$

$$\overset{(***)}{=} \left[x^{\alpha} \right]^{b}$$

$$= x^{b\alpha} = \text{same type as } x^{\alpha}$$

Hence, F also belongs to Weibull limit.

Hence, F and G belong to the same domain of attraction.

12) Suppose a class of distributions with CDF and PDF given by

$$F(x) = \frac{\beta^a}{B(a,b)} \int_{-\infty}^{x} \frac{g(t)\,[G(t)]^{a-1}\,[1-G(t)]^{b-1}}{[1-(1-\beta)\,G(t)]^{a+b}}\,dt$$

and

$$f(x) = \frac{\beta^a}{B(a,b)} \frac{g(x)\,[G(x)]^{a-1}\,[1-G(x)]^{b-1}}{[1-(1-\beta)\,G(x)]^{a+b}}$$

where $a>0$, $b>0$, $\beta>0$, $G(\cdot)$ is a valid CDF and $g(x) = \frac{d}{dx}G(x)$. Show that F and G belong to the same domain of attraction. Assume $w(F) = w(G)$.

A) Assume G belongs to Gumbel limit and show F also " " " "

B) Assume G belongs to Frēchet limit and show F also " " " "

C) Assume G belongs to Weibull limit and show F also " " " " .

A) Assume G belongs to Gumbel limit.

That is, there exists $\gamma(t) > 0$ such that

$$\lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} = e^{-x} \quad \ldots \quad (*)$$

We have

$$\lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$\underset{\text{rule}}{\overset{L'H}{=}} \lim_{t \to w(F)} \frac{-f(t + x\gamma(t))}{-f(t)} \left(1 + x\gamma'(t)\right)$$

$$= \lim_{t \to w(F)} \frac{f(t + x\gamma(t))}{f(t)} \left(1 + x\gamma'(t)\right)$$

$$= \lim_{t \to w(F)} \frac{\cancel{\frac{\beta^a}{B(a,b)}} \dfrac{g(t+x\gamma(t))\left[G(t+x\gamma(t))\right]^{a-1}\left[1 - G(t+x\gamma(t))\right]^{b-1}}{\left[1 - (1-\beta)G(t+x\gamma(t))\right]^{a+b}}}{\cancel{\frac{\beta^a}{B(a,b)}} \dfrac{g(t)\left[G(t)\right]^{a-1}\left[1 - G(t)\right]^{b-1}}{\left[1 - (1-\beta)G(t)\right]^{a+b}}}$$

$$\cdot \left(1 + x\gamma'(t)\right)$$

$$\underset{=w(G)}{\overset{w(F)}{=}} \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} \left[\frac{\underbrace{\boxed{G(t+x\gamma(t))}}_{\to 1}}{\underbrace{\boxed{G(t)}}_{\to 1}}\right]^{a-1} \left[\frac{1 - G(t+x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

$$\cdot \left[\frac{1 - (1-\beta)\underbrace{\boxed{G(t)}}_{\to 1}}{1 - (1-\beta)\underbrace{\boxed{G(t+x\gamma(t))}}_{\to 1}}\right]^{a+b} \left(1 + x\gamma'(t)\right)$$

$$= \lim_{t \to w(G)} \frac{g(t + x\gamma(t))}{g(t)} (1 + x\gamma'(t)) \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

$$\underset{\substack{\text{rule}\\\text{in}\\\text{reverse}}}{\overset{L'H}{=}} \lim_{t \to w(G)} \frac{1 - G(t + x\gamma(t))}{1 - G(t)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b-1}$$

$$= \lim_{t \to w(G)} \left[\frac{1 - G(t + x\gamma(t))}{1 - G(t)}\right]^{b}$$

$$\overset{(*)}{=} \left[e^{-x}\right]^{b}$$

$$= e^{-bx}$$

$=$ same type as e^{-x}

Hence, F also belongs to Gumbel limit

B) Assume G belongs to Fréchet limit.

That is,

$$\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\alpha}, \quad \alpha > 0 \qquad \ldots \quad (**)$$

We have

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$\overset{L'H}{\underset{\text{rule}}{=}} \lim_{t \to \infty} \frac{-x f(tx)}{-f(t)}$$

$$= \lim_{t \to \infty} \frac{x f(tx)}{f(t)}$$

$$= \lim_{t \to \infty} \frac{x \cdot \cancel{\dfrac{\beta^{\alpha}}{B(a,b)}} \dfrac{g(tx)\,[G(tx)]^{a-1}[1 - G(tx)]^{b-1}}{[1 - (1-\beta)\,G(tx)]^{a+b}}}{\cancel{\dfrac{\beta^{\alpha}}{B(a,b)}} \dfrac{g(t)\,[G(t)]^{a-1}[1 - G(t)]^{b-1}}{[1 - (1-\beta)\,G(t)]^{a+b}}}$$

$$= \lim_{t \to \infty} x \frac{g(tx)}{g(t)} \left[\frac{\overset{\to 1}{\boxed{G(tx)}}}{\underset{\to 1}{\boxed{G(t)}}}\right]^{a-1} \left[\frac{1 - G(tx)}{1 - G(t)}\right]^{b-1} \left[\frac{1 - (1-\beta)\overset{\to 1}{\boxed{G(t)}}}{1 - (1-\beta)\underset{\to 1}{\boxed{G(tx)}}}\right]^{a+b}$$

$$= \lim_{t \to \infty} \frac{x\, g(tx)}{g(t)} \left[\frac{1 - G(tx)}{1 - G(t)}\right]^{b-1}$$

$$\overset{L'H}{\underset{\text{rule in reverse}}{=}} \lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} \left[\frac{1 - G(tx)}{1 - G(t)}\right]^{b-1}$$

$$= \lim_{t \to \infty} \left[\frac{1 - G(tx)}{1 - G(t)} \right]^b$$

$\underline{\underline{(**)}}$ $\left[x^{-\alpha} \right]^b$

$= x^{-b\alpha}$

$=$ same type as $x^{-\alpha}$

Hence, F also belongs to Frēchet limit

c) Assume G belongs to Weibull limit. That is,

$$\lim_{t \to 0} \frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} = x^{\alpha}, \quad \alpha > 0 \quad \cdots \quad (***)$$

We have

$$\lim_{t \to 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)}$$

$$\overset{\text{L'H rule}}{=} \lim_{t \to 0} \frac{x f(w(F) - tx)}{f(w(F) - t)}$$

$$\overset{w(F) = w(G)}{=} \lim_{t \to 0} \frac{x f(w(G) - tx)}{f(w(G) - t)}$$

$$= \lim_{t \to 0} \frac{x \cancel{\frac{\beta^{a}}{B(a,b)}} \frac{g(w(G) - tx)[G(w(G) - tx)]^{a-1}[1 - G(w(G) - tx)]^{b-1}}{[1 - (1-\beta) G(w(G) - tx)]^{a+b}}}{\cancel{\frac{\beta^{a}}{B(a,b)}} \frac{g(w(G) - t)[G(w(G) - t)]^{a-1}[1 - G(w(G) - t)]^{b-1}}{[1 - (1-\beta) G(w(G) - t)]^{a+b}}}$$

$$= \lim_{t \to 0} x \frac{g(w(G) - tx)}{g(w(G) - t)} \left[\frac{\boxed{G(w(G) - tx)} \to 1}{\boxed{G(w(G) - t)} \to 1} \right]^{a-1} \left[\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \right]^{b-1}$$

$$\cdot \left[\frac{1 - (1-\beta) \boxed{G(w(G) - t)} \to 1}{1 - (1-\beta) \boxed{G(w(G) - tx)} \to 1} \right]^{a+b}$$

$$= \lim_{t \to 0} \frac{x\, g(w(G) - tx)}{g(w(G) - t)} \left[\frac{1 - G(w(G) - tx)}{1 - G(w(G) - t)} \right]^{b-1}$$

$$\underset{\substack{\text{rule}\\ \text{in}\\ \text{reverse}}}{\overset{\text{L'H}}{=}} \lim_{t \to 0} \frac{1-G(w(G)-tx)}{1-G(w(G)-t)} \left[\frac{1-G(w(G)-tx)}{1-G(w(G)-t)}\right]^{b-1}$$

$$= \lim_{t \to 0} \left[\frac{1-G(w(G)-tx)}{1-G(w(G)-t)}\right]^{b}$$

$$\underset{=}{(***)} \quad \left[x^{\alpha}\right]^{b}$$

$$= x^{b\alpha}$$

= same type as x^{α}

Hence, F also belongs to Weibull limit.

Hence, F and G belong to the same domain of attraction.

13)

$$G(x) = e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

Find $G^{-1}(x)$.

Set $G(x) = p$

$$\Longleftrightarrow \quad e^{-\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = p$$

$$\Longleftrightarrow \quad -\left(1+\xi \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = \log p$$

$$\Longleftrightarrow \quad 1+\xi \frac{x-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\Longleftrightarrow \quad \xi \frac{x-\mu}{\sigma} = (-\log p)^{-\xi} - 1$$

$$\Longleftrightarrow \quad x = \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right] + \mu$$

$$\Longleftrightarrow \quad G^{-1}(p) = \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right] + \mu$$

14)

$$G(x) = 1 - \left(1 + \xi \frac{x-t}{\sigma}\right)^{-\frac{1}{\xi}}$$

Find $G^{-1}(x)$.

Set $G(x) = p$

$$\Leftrightarrow 1 - \left(1 + \xi \frac{x-t}{\sigma}\right)^{-\frac{1}{\xi}} = p$$

$$\Leftrightarrow \left(1 + \xi \frac{x-t}{\sigma}\right)^{-\frac{1}{\xi}} = 1 - p$$

$$\Leftrightarrow 1 + \xi \frac{x-t}{\sigma} = (1-p)^{-\xi}$$

$$\Leftrightarrow \xi \frac{x-t}{\sigma} = (1-p)^{-\xi} - 1$$

$$\Leftrightarrow x = t + \frac{\sigma}{\xi}\left[(1-p)^{-\xi} - 1\right].$$

$$\Leftrightarrow G^{-1}(p) = t + \frac{\sigma}{\xi}\left[(1-p)^{-\xi} - 1\right].$$