MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 11

1) For the standard generalized extreme value (GEV) distribution given by the cdf

$$F(x) = \exp\left\{-(1+\xi x)^{-1/\xi}\right\}$$

(where $1 + \xi x > 0$), derive the following:

(a) the pdf,

(b) the nth moment,

(c) the mean,

(d) the variance.

2) For the standard generalized Pareto (GP) distribution given by the cdf

$$F(x) = 1 - (1+\xi x)^{-1/\xi}$$

(where $x \geq 0$ if $\xi \geq 0$ and $0 < x < -1/\xi$ if $\xi < 0$), derive the following:

(a) the pdf,

(b) the nth moment,

(c) the mean,

(d) the variance.

Q1 (a) PDF $f(x) = \frac{dF(x)}{dx}$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(b) n th moment

$$E(X^n) = \int x^n f(x)\,dx$$

$\boxed{\xi > 0} \Rightarrow -\frac{1}{\xi} < x < \infty$

$$= \int_{-\frac{1}{\xi}}^{\infty} x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$\Rightarrow 1+\xi x = y^{-\xi}$

$\Rightarrow x = \frac{y^{-\xi}-1}{\xi}$

$\Rightarrow \frac{dx}{dy} = -y^{-\xi-1}$

$x = -\frac{1}{\xi} \Rightarrow y = \infty$

$x = \infty \Rightarrow y = 0$

$$= \int_{\infty}^{0} \left(\frac{y^{-\xi}-1}{\xi}\right)^n \left(y^{-\xi}\right)^{-\frac{1}{\xi}-1} e^{-y} \left(-y^{-\xi-1}\right) dy$$

$$= \xi^{-n} \int_{0}^{\infty} (y^{-\xi}-1)^n e^{-y}\,dy$$

$$\boxed{(b+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$$

$$= \xi^{-n} \int_0^\infty \sum_{k=0}^{n} \binom{n}{k} (-1)^k (y^{-\xi})^{n-k} e^{-y}\, dy$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^\infty y^{-\xi(n-k)} e^{-y}\, dy$$

$$\boxed{\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt}$$

$$= \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma(1-\xi(n-k)).$$

$\boxed{\xi<0}$ $\Rightarrow$ $-\infty < x < -\frac{1}{\xi}$

Similar calculations show that

$$E(X^n) = \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma(1-\xi(n-k))$$

$\boxed{\xi=0}$ CDF $F(x) = e^{-e^{-x}}$

PDF $f(x) = e^{-x} e^{-e^{-x}}$

$-\infty < x < \infty$

$$E(X^n) = \int_{-\infty}^{\infty} x^n e^{-x} e^{-e^{-x}}\, dx$$

Set $y = e^{-x}$

$\Rightarrow x = -\log y$

$\Rightarrow \frac{dx}{dy} = -\frac{1}{y}$

$x = -\infty \Rightarrow y = \infty$

$x = \infty \Rightarrow y = 0$

$$= \int_{\infty}^{0} (-\log y)^n \cancel{y} e^{-y} \left(-\frac{1}{\cancel{y}}\right) dy$$

$$= (-1)^n \int_0^{\infty} (\log y)^n e^{-y} \, dy$$

$$\boxed{(\log y)^n = \frac{d^n}{da^n} y^a \Big|_{a=0}}$$

$$= (-1)^n \int_0^{\infty} \left(\frac{d^n}{da^n} y^a \Big|_{a=0}\right) e^{-y} \, dy$$

$$= (-1)^n \frac{d^n}{da^n} \left[\int_0^{\infty} y^a e^{-y} \, dy\right] \Big|_{a=0}$$

$$= (-1)^n \frac{d^n}{da^n} \Gamma(a+1) \Big|_{a=0}$$

$$E(X^n) = \begin{cases} \xi^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Gamma(1-\xi(n-k)) & \text{if } \xi \neq 0 \\ (-1)^n \frac{d^n}{da^n} \Gamma(a+1) \Big|_{a=0} & \text{if } \xi = 0 \end{cases}$$

(c) $$E(X) = \begin{cases} \frac{1}{\xi} [\Gamma(1-\xi) - 1] & \text{if } \xi \neq 0 \\ -\Gamma'(1) & \text{if } \xi = 0 \end{cases}$$

(d) $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$$= \begin{cases} \frac{1}{\xi^2}\left[\Gamma(1-2\xi) - 2\Gamma(1-\xi) + 1\right] - \frac{1}{\xi^2}\left[\Gamma(1-\xi) - 1\right]^2 & \text{if } \xi \neq 0 \\ \Gamma''(1) - [\Gamma'(1)]^2 & \text{if } \xi = 0 \end{cases}$$

Q2 $\quad F(x) = 1 - (1 + \xi x)^{-\frac{1}{\xi}}$

(a) PDF $\quad f(x) = \dfrac{dF(x)}{dx} = (1+\xi x)^{-\frac{1}{\xi}-1}$

(b) nth moment

$$= E(X^n) = \int x^n f(x)\, dx$$

$\boxed{\xi > 0} \Rightarrow 0 < x < \infty$

$$= \int_0^\infty x^n (1 + \xi x)^{-\frac{1}{\xi}-1}\, dx$$

Set $y = 1 + \xi x$

$\Rightarrow x = \dfrac{y-1}{\xi}$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{1}{\xi}$

$x = 0 \Rightarrow y = 1$

$x = \infty \Rightarrow y = \infty$

$$= \int_1^\infty \left(\frac{y-1}{\xi}\right)^n y^{-\frac{1}{\xi}-1} \frac{dy}{\xi}$$

$$= \xi^{-n-1} \int_1^\infty (y-1)^n y^{-\frac{1}{\xi}-1}\, dy$$

$$= \xi^{-n-1} \int_1^\infty \left[\sum_{k=0}^{n} \binom{n}{k} y^k (-1)^{n-k}\right] y^{-\frac{1}{\xi}-1}\, dy$$

$$= \xi^{-n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_1^\infty y^{k-\frac{1}{\xi}-1}\, dy$$

$$= \xi^{-n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left[\frac{y^{k-\frac{1}{\xi}}}{k-\frac{1}{\xi}} \right]_1^{\infty}$$

$$= \xi^{-n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{\frac{1}{\xi} - k} .$$

$\boxed{\xi < 0} \Rightarrow \quad 0 < x < -\frac{1}{\xi}$

$$E(X^n) = \int_0^{-\frac{1}{\xi}} x^n (1+\xi x)^{-\frac{1}{\xi}-1} \, dx$$

$$= \xi^{-n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{\frac{1}{\xi} - k} .$$

$\boxed{\xi = 0}$

CDF $\quad F(x) = 1 - e^{-x}$

PDF $\quad f(x) = e^{-x}$

$\quad 0 < x < \infty$

$$E(X^n) = \int_0^{\infty} x^n e^{-x} \, dx$$

$$= \Gamma(n+1)$$

$$= n!$$

$$E(X^n) = \begin{cases} -\xi^{-n-1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k}}{\frac{1}{\xi} - k} & \text{if } \xi \neq 0 \\ n! & \text{if } \xi = 0 \end{cases}$$

(c)

$$E(X) = \begin{cases} \frac{1}{1-\xi} & \text{if } \xi \neq 0 \\ 1 & \text{if } \xi = 0 \end{cases}$$

(d) $Var(X) = E(X^2) - [E(X)]^2$

$$= \begin{cases} \frac{1}{\xi^2} - \frac{2}{\xi(1-\xi)} + \frac{1}{\xi(1-2\xi)} - \left(\frac{1}{1-\xi}\right)^2 & \text{if } \xi \neq 0 \\ 2 - 1^2 = 1 & \text{if } \xi = 0 \end{cases}$$