MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 10

1) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x,y) = \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) show that the distribution is a bivariate extreme value distribution;

(b) derive the joint cdf;

(c) derive the conditional cdf if Y given $X = x$;

(d) derive the conditional cdf if X given $Y = y$;

(e) derive the joint pdf;

(f) derive the conditional pdf of Y given $X = x$;

(g) derive the conditional pdf of X given $Y = y$.

2) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x,y) = \exp\left[\frac{\alpha xy}{x+y} - x - y\right]$$

for $x > 0$ and $y > 0$.

(a) show that the distribution is a bivariate extreme value distribution;

(b) derive the joint cdf;

(c) derive the conditional cdf if Y given $X = x$;

(d) derive the conditional cdf if X given $Y = y$;

(e) derive the joint pdf;

(f) derive the conditional pdf of Y given $X = x$;

(g) derive the conditional pdf of X given $Y = y$.

3) Consider a bivariate distribution specified by the joint survival function

$$\overline{F}(x,y) = \exp\left[-\left(x^a + y^a\right)^{1/a}\right]$$

for $x > 0$ and $y > 0$.

(a) show that the distribution is a bivariate extreme value distribution;

(b) derive the joint cdf;

(c) derive the conditional cdf if Y given $X = x$;

(d) derive the conditional cdf if X given $Y = y$;

(e) derive the joint pdf;

(f) derive the conditional pdf of Y given $X = x$;

(g) derive the conditional pdf of X given $Y = y$.

1)

$$\bar{G}(x,y) = e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}$$

$x > 0$
$y > 0$
$0 < \theta < 1$

(a)

$$\begin{cases} \bar{G}(0,y) = e^{-y} \\ \bar{G}(x,0) = e^{-x} \end{cases}$$

$$\Rightarrow \left.\begin{array}{l} G_X(x) = 1 - e^{-x} \\ G_Y(y) = 1 - e^{-y} \end{array}\right\} \text{both are Exp(1) marginals}$$

$$\bar{G}(x,y) = e^{-(x+y)\left[\frac{\theta y^2}{(x+y)^2} - \frac{\theta y}{x+y} + 1\right]}$$

$$= e^{-(x+y)\left[\theta\left(\frac{y}{x+y}\right)^2 - \theta\left(\frac{y}{x+y}\right) + 1\right]}$$

$$= e^{-(x+y)\,A\left(\frac{y}{x+y}\right)}$$

where $A(w) = \theta w^2 - \theta w + 1$.
We need to show that $A(\cdot)$ satisfies

i) $A(0) = 1$

ii) $A(1) = 1$

iii) $\max(w, 1-w) \leq A(w) \leq 1 \ \forall w$

$$\Longleftrightarrow \begin{cases} (\alpha)\ A(w) \geq w \ \forall w \\ (\beta)\ A(w) \geq 1-w \ \forall w \\ (\gamma)\ A(w) \leq 1 \ \forall w \end{cases}$$

iv) $A(\cdot)$ is convex.

(i) $A(0) = \theta \cdot 0^2 - \theta \cdot 0 + 1 = 1$ ✓

(ii) $A(1) = \theta \cdot 1^2 - \theta \cdot 1 + 1 = 1$ ✓

(α) $A(\omega) \geq \omega \quad \forall \omega$

$\Leftrightarrow \theta\omega^2 - \theta\omega + 1 \geq \omega \quad \forall \omega$

$\Leftrightarrow \theta\omega(\omega - 1) + 1 - \omega \geq 0 \quad \forall \omega$

$\Leftrightarrow (1 - \theta\omega)(1 - \omega) \geq 0 \quad \forall \omega$ ✓

because $0 < \theta < 1$

(β) $A(\omega) \geq 1 - \omega \quad \forall \omega$

$\Leftrightarrow \theta\omega^2 - \theta\omega + 1 \geq 1 - \omega \quad \forall \omega$

$\Leftrightarrow \theta\omega^2 + \omega(1 - \theta) \geq 0 \quad \forall \omega$ ✓

because $0 < \theta < 1$

(γ) $A(\omega) \leq 1 \quad \forall \omega$

$\Leftrightarrow \theta\omega^2 - \theta\omega + 1 \leq 1 \quad \forall \omega$

$\Leftrightarrow \theta\omega^2 - \theta\omega \leq 0 \quad \forall \omega$

$\Leftrightarrow \theta\omega(\omega - 1) \leq 0 \quad \forall \omega$ ✓

(iv) $A(\omega) = \theta\omega^2 - \theta\omega + 1$

$A'(\omega) = 2\theta\omega - \theta$

$A''(\omega) = 2\theta > 0$

$\Rightarrow A(\cdot)$ is convex.

$\Rightarrow \bar{G}$ corresponds to a bivariate extreme value distribution.

(b) joint CDF

$$G(x, y) = 1 - \bar{G}(x, 0) - \bar{G}(0, y) + \bar{G}(x, y)$$

(c) conditional CDF of Y given $X = x$

$$\frac{\frac{\partial G(x, y)}{\partial x}}{e^{-x}}$$

(d) Conditional CDF of X given $Y = y$

$$\frac{\frac{\partial G(x, y)}{\partial y}}{e^{-y}}$$

(e) Joint PDF

$$\frac{\partial^2}{\partial x \partial y} G(x, y) = g(x, y)$$

(f) Conditional PDF of Y given $X = x$

$$\frac{g(x, y)}{e^{-x}}$$

(g) Conditional PDF of X given $Y = y$

$$\frac{g(x, y)}{e^{-y}}.$$

2) a)

$$\bar{G}(x,y) = e^{\frac{a x y}{x+y} - x - y},$$

$x > 0$
$y > 0$
$0 < a < 1$

$$\bar{G}(0,y) = e^{-y}$$
$$\bar{G}(x,0) = e^{-x}$$

$$\Rightarrow \left.\begin{array}{l} G_X(x) = 1 - e^{-x} \\ G_Y(y) = 1 - e^{-y} \end{array}\right\} \begin{array}{l} Exp(1) \\ \text{marginals} \end{array}$$

$$\bar{G}(x,y) = e^{-(x+y)\left[-\frac{a x y}{(x+y)^2} + 1\right]}$$

$$= e^{-(x+y)\left[-a\left(\frac{y}{x+y}\right)\left(1-\frac{y}{x+y}\right)+1\right]}$$

$$= e^{-(x+y)\, A\left(\frac{y}{x+y}\right)}$$

where $A(w) = -a w(1-w) + 1$

(i) $A(0) = -a \cdot 0 \cdot (1-0) + 1 = 1$ ✓

(ii) $A(1) = -a \cdot 1 \cdot (1-1) + 1 = 1$ ✓

(α) $A(w) \geq w \;\forall w$

$\Leftrightarrow -a w(1-w) + 1 \geq w \;\forall w$

$\Leftrightarrow (1-w)(1-aw) \geq 0 \;\forall w$ ✓

because $0 < a < 1$

(β) $A(w) \geq 1-w \;\forall w$

$\Leftrightarrow 1 - a w(1-w) \geq 1 - w \;\forall w$

$\Leftrightarrow w[1 - a(1-w)] \geq 0 \;\forall w$ ✓

because $0 < a < 1$

(γ) $A(\omega) \leq 1 \ \forall \ \omega$

$\Leftrightarrow \ 1 - a\omega(1-\omega) \leq 1 \ \forall \ \omega$

$\Leftrightarrow \ -a\omega(1-\omega) \leq 0 \ \forall \ \omega$ ✓

(iv) $A(\omega) = 1 - \alpha\omega(1-\omega)$

$A'(\omega) = -\alpha + 2\alpha\omega$

$A''(\omega) = 2\alpha > 0$

$\Rightarrow A(\cdot)$ is convex.

Hence, $\bar{G}(x, y)$ corresponds to a bivariate extreme value distribution.

The remaining parts are similar to Q1.

3) $$\bar{G}(x,y) = e^{-(x^a+y^a)^{\frac{1}{a}}}, \quad a>0,\ x>0,\ y>0$$

$$\bar{G}(0,y) = e^{-y}$$
$$\bar{G}(x,0) = e^{-x}$$

$$\Rightarrow \left.\begin{array}{l} G_X(x) = 1-e^{-x} \\ G_Y(y) = 1-e^{-y} \end{array}\right\} \text{Exp}(1) \text{ marginals}$$

$$\bar{G}(x,y) = e^{-(x+y)\left[\left(\frac{x}{x+y}\right)^a + \left(\frac{y}{x+y}\right)^a\right]^{\frac{1}{a}}}$$
$$= e^{-(x+y)\,A\left(\frac{y}{x+y}\right)}$$

where $A(w) = \left[(1-w)^a + w^a\right]^{\frac{1}{a}}$

(i) $A(0) = \left[(1-0)^a + 0^a\right]^{\frac{1}{a}} = 1$ ✓

(ii) $A(1) = \left[(1-1)^a + 1^a\right]^{\frac{1}{a}} = 1$ ✓

(α) $A(w) \geq w \ \forall w$

$$\Leftrightarrow \left[w^a + (1-w)^a\right]^{\frac{1}{a}} \geq w \ \forall w$$
$$\Leftrightarrow \cancel{w^a} + (1-w)^a \geq \cancel{w^a} \ \forall w$$
$$\Leftrightarrow (1-w)^a \geq 0 \ \forall w \ ✓$$

(β) $A(w) \geq 1-w \ \forall w$

$$\Leftrightarrow \left[w^a + (1-w)^a\right]^{\frac{1}{a}} \geq 1-w \ \forall w$$
$$\Leftrightarrow w^a + \cancel{(1-w)^a} \geq \cancel{(1-w)^a} \ \forall w$$
$$\Leftrightarrow w^a \geq 0 \ \forall w \ ✓$$

(γ) $\omega \leq 1$

$\Rightarrow \omega^a \leq \omega$ assuming $\underline{a \geq 1}$

$\Rightarrow (1-\omega)^a \leq 1-\omega$

$\Rightarrow \omega^a + (1-\omega)^a \leq 1 \quad \forall \omega$

$\Rightarrow \left[\omega^a + (1-\omega)^a\right]^{\frac{1}{a}} \leq 1 \quad \forall \omega$

$\Rightarrow A(\omega) \leq 1 \quad \forall \omega$

(iv) $A(\omega) = \left[\omega^a + (1-\omega)^a\right]^{\frac{1}{a}}$

$A'(\omega) = \left[\omega^a + (1-\omega)^a\right]^{\frac{1}{a}-1} \left(\omega^{a-1} - (1-\omega)^{a-1}\right)$

$$A''(\omega) = \underbrace{\boxed{(a-1)}}_{>0 \text{ if } a \geq 1} \boxed{\left[\omega^a + (1-\omega)^a\right]^{\frac{1}{a}-2}} > 0$$

$$\cdot \left\{ \underbrace{\boxed{\left[\omega^{a-1} - (1-\omega)^{a-1}\right]^2}}_{\geq 0} + \underbrace{\boxed{\left[\omega^a + (1-\omega)^a\right]}}_{>0} \underbrace{\boxed{\left[\omega^{a-2} + (1-\omega)^{a-2}\right]}}_{>0} \right\}$$

$> 0 \; \forall \omega$

$\Rightarrow A(\cdot)$ is convex.

Hence, $\bar{G}$ corresponds to a bivariate extreme value distribution.

The remaing parts are similar to Q1.