MATH4/68181: Extreme values and financial risk
Semester 1
Problem sheet for Week 1

1. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = \sigma^{-1} \exp\left(-\frac{1}{\sigma}x\right) \exp\left\{-\exp\left(-\frac{x}{\sigma}\right)\right\}$$

find the mle of σ.

2. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = \lambda \sigma^{\lambda} x^{-\lambda-1} \exp\left(-\sigma^{\lambda} x^{-\lambda}\right)$$

find the mles of λ and σ.

3. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = \lambda \sigma^{-\lambda} x^{\lambda-1} \exp\left(-\sigma^{-\lambda} x^{\lambda}\right)$$

find the mles of λ and σ.

4. If $x_1, x_2, \ldots, x_n$ is a random sample from

$$f(x) = (1 - \lambda x)^{1/\lambda - 1}$$

find the mle of λ.

1) Suppose $x_1, \ldots, x_n$ IID from

$$f(x) = \frac{1}{\sigma} e^{-\frac{x}{\sigma}} e^{-e^{-\frac{x}{\sigma}}}.$$

The likelihood function is

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}.$$

The log-likelihood function is

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

The derivative wrt σ is

$$\frac{d \log L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}}$$

The MLE of σ is the root of

$$\frac{d \log L(\sigma)}{d\sigma} = 0$$

$$\Longleftrightarrow \quad n\sigma = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}}$$

This must be solved numerically for σ.

2) Suppose $x_1, \ldots, x_n$ IID with PDF

$$f(x) = \lambda \sigma^{\lambda} x^{-\lambda-1} e^{-\sigma^{\lambda} x^{-\lambda}}.$$

The likelihood function

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\sigma^{\lambda} x_i^{-\lambda}} \right]$$

$$= \lambda^{n} \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}}.$$

The log-likelihood function is

$$\log L(\lambda, \sigma) = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^{n} \log x_i - \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}.$$

The partial derivative wrt λ and σ are

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right) = 0 \text{ —— } (1)$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \lambda \sigma^{\lambda-1} \sum_{i=1}^{n} x_i^{-\lambda} = 0 \text{ —— } (2)$$

$$(2) \Rightarrow \quad \frac{n}{\sigma^{\lambda}} = \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\Rightarrow \quad \sigma = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right]^{\frac{1}{\lambda}} \qquad \text{——— (3)}$$

Substituting (3) into (1) gives

$$\frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i$$

$$- \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}}\right] \sum_{i=1}^{n} x_i^{-\lambda} \left[\log \sigma - \log x_i\right] = 0$$

$$\Leftrightarrow \quad \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i$$

$$- \left[\frac{n}{\cancel{\sum_{i=1}^{n} x_i^{-\lambda}}}\right] \left[\cancel{\sum_{i=1}^{n} x_i^{-\lambda}}\right] \log \sigma$$

$$+ \frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i = 0$$

$$\Leftrightarrow \quad \frac{n}{\lambda} + \cancel{n \log \sigma} - \sum_{i=1}^{n} \log x_i - \cancel{n \log \sigma}$$

$$+ \frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i = 0$$

$$\Leftrightarrow \quad \frac{n}{\lambda} = \sum_{i=1}^{n} \log x_i - n \frac{\sum_{i=1}^{n} x_i^{-\lambda} \log x_i}{\sum_{i=1}^{n} x_i^{-\lambda}} \qquad \text{——— (4)}$$

Solve (4) to get $\hat{\lambda}$. Substitute $\hat{\lambda}$ into (3) to

3) Suppose $x_1, \ldots, x_n$ are IID with PDF

$$f(x) = \lambda \sigma^{-\lambda} x^{\lambda-1} e^{-\sigma^{-\lambda} x^{\lambda}}.$$

The likelihood function is

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\sigma^{-\lambda} x_i^{\lambda}} \right]$$

$$= \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} e^{-\sigma^{-\lambda} \sum_{i=1}^{n} x_i^{\lambda}}.$$

The log-likelihood is

$$\log L(\lambda, \sigma) = n \log \lambda - n\lambda \log \sigma + (\lambda-1) \sum_{i=1}^{n} \log x_i - \sigma^{-\lambda} \sum_{i=1}^{n} x_i^{\lambda}.$$

The partial derivatives wrt λ and σ are

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log\left(\frac{x_i}{\sigma}\right) = 0 \quad \text{—— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sigma^{-\lambda-1} \sum_{i=1}^{n} x_i^{\lambda} = 0 \quad \text{— (2)}$$

$$(2) \Rightarrow \quad n\sigma^{\lambda} = \sum_{i=1}^{n} x_i^{\lambda}$$

$$\Rightarrow \quad \sigma = \left[\frac{\sum_{i=1}^{n} x_i^{\lambda}}{n} \right]^{\frac{1}{\lambda}} \qquad \text{———} \quad (3)$$

Substituting (3) into (1) gives

$$\frac{n}{\lambda} - n\log\sigma + \sum_{i=1}^{n} \log x_i - \left[\frac{n}{\sum_{i=1}^{n} x_i^{\lambda}}\right] \sum_{i=1}^{n} x_i^{\lambda}\left[\log x_i - \log\sigma\right]$$

$$= 0$$

$$\Leftrightarrow \frac{n}{\lambda} - n\log\sigma + \sum_{i=1}^{n} \log x_i - \left[\frac{n}{\sum_{i=1}^{n} x_i^{\lambda}}\right] \sum_{i=1}^{n} x_i^{\lambda} \log x_i$$

$$+ \left[\frac{n}{\cancel{\sum_{i=1}^{n} x_i^{\lambda}}}\right] \left[\cancel{\sum_{i=1}^{n} x_i^{\lambda}}\right] \log\sigma = 0$$

$$\Leftrightarrow \frac{n}{\lambda} - \cancel{n\log\sigma} + \sum_{i=1}^{n} \log x_i - n\,\frac{\sum_{i=1}^{n} x_i^{\lambda} \log x_i}{\sum_{i=1}^{n} x_i^{\lambda}} + \cancel{n\log\sigma} = 0$$

$$\Leftrightarrow \frac{n}{\lambda} + \sum_{i=1}^{n} \log x_i = n\,\frac{\sum_{i=1}^{n} x_i^{\lambda} \log x_i}{\sum_{i=1}^{n} x_i^{\lambda}} = 0 \quad \text{——} \quad (4)$$

Solve (4) to get $\hat{\lambda}$. Substitute $\hat{\lambda}$ into (3) to get $\hat{\sigma}$.

4) Suppose $x_1, \ldots, x_n$ IID with PDF

$$f(x) = (1-\lambda x)^{\frac{1}{\lambda}-1}.$$

The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} (1-\lambda x_i)^{\frac{1}{\lambda}-1}$$

$$= \left[\prod_{i=1}^{n} (1-\lambda x_i)\right]^{\frac{1}{\lambda}-1}.$$

The log-likelihood is

$$\log L(\lambda) = \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \log(1-\lambda x_i).$$

The derivative wrt λ is

$$\frac{d \log L(\lambda)}{d\lambda} = -\lambda^{-2} \sum_{i=1}^{n} \log(1-\lambda x_i) - \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \frac{x_i}{1-\lambda x_i}.$$

The MLE of λ is to root of

$$\frac{d \log L(\lambda)}{d\lambda} = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \log(1-\lambda x_i) = \lambda(\lambda-1) \sum_{i=1}^{n} \frac{x_i}{1-\lambda x_i}.$$

This must be solved for λ numerically