MATH10282 Introduction to Statistics
Semester 2, 2019/2020
Example Sheet 11 - Solutions

1. In both cases an appropriate test statistic is

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\,,$$

with $\mu_0 = 200$, $\sigma = 30$ and $n = 40$. Under $H_0 : \mu = 200$, $Z \sim N(0, 1)$.

(i) In this case the rejection region is $|Z| > z_{1-\alpha/2}$. For $\alpha = 0.05$ and $\alpha = 0.01$, the critical values are 1.960 and 2.576 respectively.

(ii) In this case the rejection region is $Z > z_{1-\alpha}$. For $\alpha = 0.05$ and $\alpha = 0.01$, the critical values are 1.645 and 2.326 respectively.

Here $\bar{x} = \sum_{i=1}^{40} x_i/40 = 8328.4/40 = 208.21$, and so $Z = 1.731$.

(i) H_0 is not rejected in favour of the two-sided alternative $H_1 : \mu \neq \mu_0$ at either the 5% or 1% significance level. In this case we conclude there is insufficient evidence to reject the claim that $\mu = 200$.

(ii) H_0 is rejected in favour of the one-sided alternative $H_1 : \mu > \mu_0$ at the 5% significance level but not at the 1% level.

2. (i) The test statistic is

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{0.7 - 0}{\sqrt{3/10}} = 1.278\,.$$

At the 5% significance level the rejection region is $Z > z_{0.95} = 1.645$. Hence, there is insufficient evidence to reject the claim that $\mu = 0$ in favour of $H_1 : \mu > 0$.

(ii) Using the above test H_0 is rejected if $\bar{x} > 0 + 1.645\sqrt{3/10} = 0.901$. The probability of a type II error is

$$\begin{aligned}
\mathrm{P}(\text{do not reject } H_0 \,|\, H_1) &= \mathrm{P}\left(\bar{X} \leq 0.901 \,|\, H_1\right) \\
&= \mathrm{P}\left(\frac{\bar{X} - \mu}{\sqrt{3/10}} \leq \frac{1.645\sqrt{3/10} - \mu}{\sqrt{3/10}}\right) \\
&= \Phi\left(1.645 - \frac{\mu}{\sqrt{3/10}}\right) \\
&\quad \text{since } \frac{\bar{X} - \mu}{\sqrt{3/10}} \sim N(0, 1) \text{ under } H_1.
\end{aligned}$$

For $\mu = 1$ the probability of a type II error is $\Phi(-0.181) = 0.43$. For $\mu = 1.5$ the probability of a type II error is $\Phi(-1.09) = 0.14$. As the value of μ increases, the probability of a type I error decreases.

(iii) With the same sample mean, H_0 would be rejected if

$$\frac{0.7\sqrt{n}}{\sqrt{3}} > 1.645 \iff n > 1.645^2 \times 3/0.7^2 = 16.5675.$$

Hence n needs to be at least 17 for H_0 to be rejected with the same value of $\bar{x}$.

3. We are not told that the distribution is normal, hence we use the test for non-normal data. An appropriate test statistic is

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}\,.$$

Under H_0, as n is large Z is approximately distributed as $N(0,1)$. Thus an appropriate rejection region at the approximate 5% level is to reject H_0 if $|Z| > z_{0.975} = 1.96$. The observed value of Z is

$$z = \frac{1794.6 - 1800}{\sqrt{2484/250}} = -1.713\,.$$

As $z > -1.96$, we do not reject H_0. The conclusion is that there is insufficient evidence to reject the claim that $\mu = 1800$.

4. (i) Assuming the data are normally distributed, an appropriate test statistic for testing $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$ is

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\,.$$

Under H_0, $Z \sim N(0,1)$. At the 5% significance level, we reject H_0 in favour of H_1 if $Z > z_{0.95}$, where $z_{0.95} = 1.645$ is the 0.95 point of a $N(0,1)$ distribution.

Here the observed value of Z is $z = \frac{252.96-250}{8.7/\sqrt{20}} = 1.521$. This is less than $z_{0.95}$ and so we conclude that there is insufficient evidence to reject the claim that $\mu = 250$.

(ii) With the above test, we reject H_0 if $\bar{X} > 250 + 1.645 \times 8.7/\sqrt{20} = 253.2$. The probability of rejecting H_0 is

$$\mathrm{P}\left(\bar{X} > 253.2\right) = \mathrm{P}\left(\frac{\bar{X} - \mu}{8.7/\sqrt{20}} > \frac{253.2 - \mu}{8.7/\sqrt{20}}\right) = 1 - \Phi\left(\frac{253.2 - \mu}{8.7/\sqrt{20}}\right)\,.$$

Thus the probability of rejecting H_0 is 0.82 if $\mu = 255$ and 0.98 if $\mu = 257$. The probability of rejecting H_0 increases as μ increases above 250.

5. Assuming the data are normally distributed with unknown mean and variance, an appropriate test statistic for testing $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$ is

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}\,.$$

At the 5% significance level, we reject H_0 if $T > t_{0.975}$ or $T < -t_{0.975}$, where $t_{0.975} = 2.006$ is the 0.95 point of a Student t distribution on $n - 1 = 53$ degrees of freedom. Here, the observed value of T is

$$t = \frac{66.3 - 64}{9.2/\sqrt{54}} = 1.837\,.$$

Thus, t is between $-t_{0.975}$ and $t_{0.975}$ and so H_0 is not rejected at the 5% level. We conclude that there is insufficient evidence at the 5% significance level to reject the claim that the mean mark in the school is equal to 64.

6. (i) A suitable test statistic is

$$Y = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}}\,,$$

where $\hat{p} = 478/920 = 0.5196$ is the sample proportion who are concerned about the issue, $p_0 = 0.5$ is the hypothesized value under H_0, and $n = 920$ is the sample size.

(ii) Under H_0, since $n = 920$ is large, $Y \sim N(0, 1)$ approximately. To achieve an approximate significance level of 5%, an appropriate critical region is

$$Y > z_{0.95}\,,$$

where the critical value $z_{0.95} = 1.645$ is the 0.95 point of a $N(0, 1)$ distribution.

(iii) We have $Y = 1.1869$, which is less than the critical value. Thus there is insufficient evidence to reject the null hypothesis that $p_0 = 0.5$.

(iv) The procedure rejects H_0 if

$$\frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} > 1.645\,,$$

i.e. if

$$\hat{p} > 0.5 + 1.645 \times \sqrt{0.5 \times 0.5/920} = 0.5271\,.$$

If the true value of $p = 0.55$, the probability of H_0 being rejected is

$$\begin{aligned}\mathrm{P}(\hat{p} > 0.5271) &= \mathrm{P}\left(\frac{\hat{p} - 0.55}{\sqrt{0.55 \times 0.45/920}} > \frac{0.5271 - 0.55}{\sqrt{0.55 \times 0.45/920}}\right)\\ &\approx 1 - \Phi(-1.396) = 0.92\,.\end{aligned}$$

7. (i) Let X be the number of left handed people in a random sample of size $n = 400$. We have that $X \sim \text{Bi}(n, p)$, where p is the proportion of left-handers in the population.

An appropriate test statistic for testing $H_0 : p = p_0$ vs $H_1 : p \neq p_0$ is

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}},$$

where $\hat{p} = X/n$ is the sample proportion of left-handers. Here, $p_0 = 0.1$.

Under H_0, as $400 = n \geq 9 \max\{p_0/(1-p_0), (1-p_0)/p_0\} = 81$, we have that Z is approximately distributed as $N(0, 1)$. Thus, to achieve an approximate significance level of 5%, an appropriate rejection region is to reject H_0 if $|Z| > z_{0.975}$, where $z_{0.975} = 1.96$ is the 0.975 point of a $N(0, 1)$ distribution.

Here, $\hat{p} = 47/400 = 0.1175$ and so the observed value of Z is

$$z = \frac{0.1175 - 0.1}{\sqrt{0.1 \times 0.9/400}} = 1.166667\,.$$

Hence $|z| \leq z_{0.975}$ and we do not reject H_0. Thus there is insufficient evidence at the 5% level to reject the claim that the proportion of left handed people in the population is equal to 10%.

(ii) Above, H_0 is rejected if $\hat{p} > p_0 + 1.96\sqrt{p_0(1 - p_0)/n} = 0.1294$ or if $\hat{p} < p_0 - 1.96\sqrt{p_0(1 - p_0)/n} = 0.0706$. Thus, the probability of rejecting H_0 is

$$\begin{aligned}
&1 - \text{P}(0.0706 < \hat{p} < 0.1294) \\
&= 1 - \text{P}\left(\frac{0.0706 - p}{\sqrt{p(1-p)/n}} \leq \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} \leq \frac{0.1294 - p}{\sqrt{p(1-p)/n}}\right) \\
&\approx 1 - \left[\Phi\left(\frac{0.1294 - p}{\sqrt{p(1-p)/n}}\right) - \Phi\left(\frac{0.0706 - p}{\sqrt{p(1-p)/n}}\right)\right],
\end{aligned}$$

since, as n is large, $\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \sim N(0, 1)$ approximately.

When $p = 0.103$, the approximate probability of rejecting H_0 is $1-\Phi(1.737)+\Phi(-2.132) = 0.058$. When $p = 0.105$, the approximate probability of rejecting H_0 is $1 - \Phi(1.592) + \Phi(-2.244) = 0.068$.

For both of these values of p, the probability of rejecting H_0 is quite small since they are both close to p_0. However, the probability does increase as $|p - p_0|$ increases.