MATH10282 Introduction to Statistics
Semester 2, 2019/2020
Example Sheet 6 - Solutions

1. (i) Assuming that each passenger has the same probability of showing up, and each passenger shows up or not independently of all other passengers, then $Y \sim \mathrm{Bi}(n, p)$ with $n = 267$ and $p = 1 - 0.06 = 0.94$. In this case, $\mathrm{E}(Y) = np = 250.98$ and $\mathrm{Var}(Y) = np(1-p) = 15.0588$.

(ii) Recall from the notes that the normal approximation is valid if

$$n \geq 9 \max\left\{\frac{1-p}{p}, \frac{p}{1-p}\right\}.$$

In this case, $9p/(1-p) = 141$ and $9(1-p)/p = 0.57$ and $n = 267$. Thus the normal approximation is valid.

(iii) Using a continuity correction, the approximate probability that between 248 and 255 passengers show up is

$$\begin{aligned}
\mathrm{P}(248 \leq Y \leq 255) &\approx \mathrm{P}\left(247.5 \leq X \leq 255.5\right) \\
&\qquad \text{where } X \sim N(250.98, 15.0588) \\
&= \mathrm{P}\left(\frac{247.5 - 250.98}{\sqrt{15.0588}} \leq Z \leq \frac{255.5 - 250.98}{\sqrt{15.0588}}\right), \quad Z \sim N(0,1) \\
&= \Phi(1.165) - \Phi(-0.897) \\
&= 0.8780 - 0.1849 = 0.6931\,.
\end{aligned}$$

However, if more than 255 passengers arrive then the plane will take off full and the remaining passengers will be transferred to another flight. The probability that the flight takes off with between 248 and 255 passengers is therefore

$$\begin{aligned}
\mathrm{P}\left(Y \geq 248\right) &\approx \mathrm{P}\left(X \geq 247.5\right) = 1 - \mathrm{P}(X \leq 247.5) \\
&\approx 1 - \Phi(-0.897) = 0.8151\,.
\end{aligned}$$

(iv) The probability that there will be a seat for all passengers who show up is

$$\begin{aligned}
\mathrm{P}(Y \leq 255) &\approx \mathrm{P}(X \leq 255.5) = \mathrm{P}\left(Z \leq \frac{255.5 - 250.98}{\sqrt{15.0588}}\right) \\
&= \Phi(1.165) = 0.8780
\end{aligned}$$

(v) The desired probability is equal to

$$\mathrm{P}(0.93 \leq Y/267 \leq 0.94)\,.$$

Note that the distribution of $Y/267$ is approximately normal with mean $250.98/267 = 0.94$ and variance $\mathrm{Var}(Y/267) = \mathrm{Var}(Y)/(267)^2 = 0.94 \times 0.06/267 = 0.000211236$. Thus the above probability is approximately

$$\Phi\left(\frac{0.94-0.94}{\sqrt{0.000211236}}\right) - \Phi\left(\frac{0.93-0.94}{\sqrt{0.000211236}}\right) = \Phi(0) - \Phi(-0.6880)$$
$$= 0.5 - 0.2457 = 0.2543.$$

2. (i) Since $n \geq 9\max\{\frac{p}{1-p}, \frac{1-p}{p}\}$, we have that both

$$n \geq 9\left(\frac{1-p}{p}\right) \quad (*) \qquad \text{and} \qquad n \geq 9\left(\frac{p}{1-p}\right) \quad (**)\,.$$

Note also that

$$\frac{n-np}{\sqrt{np(1-p)}} = \frac{n}{\sqrt{n}}\frac{(1-p)}{\sqrt{1-p}} = \sqrt{n}\sqrt{\frac{1-p}{p}} \quad (\dagger)\,.$$

Taking square roots of $(**)$, we have that $\sqrt{n} \geq 3\sqrt{\frac{p}{1-p}}$. Substituting this into $(\dagger)$ we have

$$\frac{n-np}{\sqrt{np(1-p)}} \geq 3\sqrt{\frac{p}{1-p}}\sqrt{\frac{1-p}{p}} = 3\,,$$

thus the first statement is proved. Similarly,

$$\frac{-np}{\sqrt{np(1-p)}} = -\sqrt{n}\sqrt{\frac{p}{1-p}} \quad (\dagger\dagger)$$

Taking square roots of $(*)$ and multiplying by -1 (which reverses the inequality sign), we have $-\sqrt{n} \leq -3\sqrt{\frac{1-p}{p}}$. Substituting this into $(\dagger\dagger)$, we have

$$\frac{-np}{\sqrt{np(1-p)}} \leq -3\sqrt{\frac{1-p}{p}}\sqrt{\frac{p}{1-p}} = -3\,.$$

This proves the second statement.

(ii)

$$\mathrm{P}(0 \leq Y \leq n) = \mathrm{P}\left(\frac{-np}{\sqrt{np(1-p)}} \leq \frac{Y-np}{\sqrt{np(1-p)}} \leq \frac{n-np}{\sqrt{np(1-p)}}\right)$$
$$= \mathrm{P}\left(\frac{-np}{\sqrt{np(1-p)}} \leq Z \leq \frac{n-np}{\sqrt{np(1-p)}}\right)$$

By part (i), if $n \geq 9 \max\{\frac{p}{1-p}, \frac{1-p}{p}\}$ then $\frac{-np}{\sqrt{np(1-p)}} \leq -3$ and $\frac{n-np}{\sqrt{np(1-p)}} \geq 3$. Hence,

$$\begin{aligned}
\mathrm{P}(0 \leq Y \leq n) &= \mathrm{P}\left(\frac{-np}{\sqrt{np(1-p)}} \leq Z \leq \frac{n-np}{\sqrt{np(1-p)}}\right) \\
&\geq \mathrm{P}(-3 \leq Z \leq 3) \qquad \text{(If you are not sure why, draw a picture!)} \\
&= \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 0.9973.
\end{aligned}$$

Thus, as claimed, $\mathrm{P}(0 \leq Y \leq n) \geq 0.9973$ if $n \geq 9 \max\{\frac{p}{1-p}, \frac{1-p}{p}\}$.

(iii) Rule-of-thumb: approximation is valid if $n \geq 9 \max\left\{\frac{p}{1-p}, \frac{1-p}{p}\right\}$. If $p = 0.1$ then we need

$$n \geq 9 \max\left\{\frac{0.1}{0.9}, \frac{0.9}{1}\right\} = 9 \max\left\{\frac{1}{9}, 9\right\} = 9 \times 9 = 81\,.$$

If $n = 100$, then we need

$$\begin{aligned}
& 100 \geq 9\left(\frac{1-p}{p}\right) \qquad \text{and} \qquad 100 \geq 9\left(\frac{p}{1-p}\right) \\
\iff & 109p \geq 9 \qquad \text{and} \qquad 100 \geq 109p \\
\iff & p \in \left[\frac{9}{109}, \frac{100}{109}\right] \approx [0.0826, 0.9174]\,.
\end{aligned}$$

3. (i) $\bar{X}_{25} \sim N(112, \frac{12^2}{25})$, i.e. $N(112, 5.76)$.

(ii) $\mathrm{P}(X_{25} > 115) = 1 - \Phi\left(\frac{115-112}{\sqrt{5.76}}\right) = 1 - \Phi(1.25) = 1 - 0.944 = 0.1056$.

(iii) Observe that

$$\begin{aligned}
\mathrm{P}(-1 + \mu < \bar{X}_{25} < \mu + 1) &= \mathrm{P}\left(\frac{-1}{\sqrt{5.76}} < \frac{\bar{X}_{25} - \mu}{\sqrt{5.76}} < \frac{1}{\sqrt{5.76}}\right) \\
&= \Phi(0.417) - \Phi(-0.417) \\
&= 0.6617 - 0.3383 = 0.3234.
\end{aligned}$$

4. Let $X_1, \ldots, X_{10}$ be an independent random sample from $N(20, 3)$, and $Y_1, \ldots, Y_{15}$ be an independent random sample from the same distribution, independent of the first sample. We require $\mathrm{P}(|\bar{X}_{10} - \bar{Y}_{15}| > 0.3)$. Observe

$$\begin{aligned}
\mathrm{P}(|\bar{X}_{10} - \bar{Y}_{15}| > 0.3) &= 1 - \mathrm{P}(|\bar{X}_{10} - \bar{Y}_{15}| < 0.3) \\
&= 1 - \mathrm{P}(-0.3 < \bar{X}_{10} - \bar{Y}_{15} < 0.3)\,.
\end{aligned}$$

Note that $\bar{X}_{10} \sim N(20, 3/10)$ and $\bar{Y}_{15} \sim N(20, 3/15)$ independently, so that

$$\begin{aligned}\bar{X}_{10} - \bar{Y}_{15} &\sim N\left(0, \frac{3}{10} + \frac{3}{15}\right) \\ &\sim N(0, 1/2)\end{aligned}$$

Thus the required probability is

$$\begin{aligned}1 - \Phi\left(\frac{0.3 - 0}{\sqrt{1/2}}\right) + \Phi\left(\frac{-0.3 - 0}{\sqrt{1/2}}\right) &= 1 - \Phi(0.424) + \Phi(-0.424) \\ = 1 - 0.6642 + 0.3358 = 0.6716.\end{aligned}$$

5. Let X be the number of voters supporting Candidate A in the first sample, and let Y be the number supporting Candidate A in the second sample. Then $X \sim \mathrm{Bi}(200, 0.65)$ and $Y \sim \mathrm{Bi}(200, 0.65)$ independently.

We require

$$P\left(\left|\frac{X}{200} - \frac{Y}{200}\right| < 0.1\right) = \mathrm{P}\left(-0.1 < \frac{X}{200} - \frac{Y}{200} < 0.1\right).$$

Now, using the normal approximation to the binomial distribution, we have that $\frac{X}{200} \sim N(0.65, \frac{0.65\times 0.35}{200})$ approximately and $\frac{Y}{200} \sim N(0.65, \frac{0.65\times 0.35}{200})$ approximately. X and Y are also independent. Thus we have that, approximately,

$$\begin{aligned}\frac{X}{200} - \frac{Y}{200} &\sim N\left(0.65 - 0.65, \frac{0.65 \times 0.35}{200} + \frac{0.65 \times 0.35}{200}\right). \\ &\sim N(0, 0.002275)\end{aligned}$$

Thus the required probability is approximately

$$\begin{aligned}\Phi\left(\frac{0.1}{\sqrt{0.002275}}\right) - \Phi\left(\frac{-0.1}{\sqrt{0.002275}}\right) &= \Phi(2.097) - \Phi(-2.097) \\ = 0.9820 - 0.0180 = 0.9640\,.\end{aligned}$$

6. (i) Let $X_1, \ldots, X_n$ be an independent sample from a $\mathrm{Po}(\lambda)$ distribution, with λ unknown. Consider $\hat{\lambda} = \bar{X}_n$. We have

$$\begin{aligned}\mathrm{E}(\hat{\lambda}) &= \mathrm{E}(\bar{X}_n) = \frac{n\lambda}{n} = \lambda\,. \\ \mathrm{Var}(\hat{\lambda}) &= \mathrm{Var}(\bar{X}_n) = \frac{n\lambda}{n^2} = \frac{\lambda}{n}\,.\end{aligned}$$

By the CLT, $\hat{\lambda} = \bar{X}_n \sim N(\lambda, \lambda/n)$ approximately for large n.

(ii) Note that $\bar{X}_{100} \sim N(36, 36/100)$ approximately, and so

$$\begin{aligned}
\mathrm{P}(35.0 < \bar{X}_{100} < 37.0) &\approx \Phi\left(\frac{37-36}{0.6}\right) - \Phi\left(\frac{35-36}{0.6}\right) \\
&= \Phi(1.667) - \Phi(-1.667) \\
&= 0.9522 - 0.0478 = 0.9044\,.
\end{aligned}$$

7. Let $X_1, \ldots, X_{10}$ be a random sample from $N(48, 36)$. Let

$$S^2 = \frac{1}{9}\sum_{i=1}^{10}(X_i - \bar{X})^2$$

be the sample variance. We have that

$$\frac{9 \times S^2}{36} \sim \chi^2(9)\,.$$

(i) Observe that

$$\begin{aligned}
\mathrm{P}(25 \le S^2 \le 60) &= \mathrm{P}\left(\frac{9 \times 25}{36} \le \frac{9 \times S^2}{36} \le \frac{9 \times 60}{36}\right) \\
&= F_Y(15) - F_Y(6.25) = 0.9090 - 0.2853 = 0.6237
\end{aligned}$$

where F_Y is the c.d.f. of a $\chi^2(9)$ random variable. Note that the χ^2 probabilities have been computed using R via `pchisq(15, df=9)` etc.

(ii) We require n such that $\mathrm{P}(S^2 > 20) = 0.9$, i.e. such that

$$\mathrm{P}\left(\frac{(n-1)S^2}{36} > \frac{(n-1) \times 20}{36}\right) = 0.9\,.$$

This is equivalent to $1 - F_Y(0.555 \times (n-1)) = 0.9$, or alternatively $F_Y(0.555 \times (n-1)) = 0.1$ where $Y \sim \chi^2(n-1)$.
By trial and error, we find that

$$\begin{aligned}
&\text{when } n-1 = 12\,, \quad F_Y(0.555 \times (n-1)) = 0.1208 \\
&\text{when } n-1 = 13\,, \quad F_Y(0.555 \times (n-1)) = 0.1093 \\
&\text{when } n-1 = 14\,, \quad F_Y(0.555 \times (n-1)) = 0.0990
\end{aligned}$$

Therefore choose $n-1 = 14$, i.e. $n = 15$, as the probability 0.0990 is closest to 0.1.

8. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with mean $\mu = 0$ and variance σ^2 which is unknown. We have

$$\hat{\sigma}^2 = k\sum_{i=1}^{n} X_i^2\,.$$

We know $\mathrm{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = \sigma^2$, for $i = 1, \dots, n$. Moreover, $\mathrm{Var}(X_i) = \mathrm{E}(X_i^2) - \mathrm{E}(X_i)^2 = \mathrm{E}(X_i^2)$. Thus,

$$\mathrm{E}(\hat{\sigma}^2) = k \sum_{i=1}^{n} \mathrm{E}(X_i^2) = kn\sigma^2 .$$

Hence $\mathrm{E}(\hat{\sigma}^2) = kn\sigma^2$, which equals σ^2 when $k = 1/n$. If $X_1, \dots, X_n \sim N(0, \sigma^2)$, the estimator $\hat{\sigma}^2$ can be scaled to give a random variable with a χ^2 distribution:

$$\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n) .$$

9. $X_1, \dots, X_n$ are a random sample from $U(0, \theta)$. We have that $\mathrm{E}(X_i) = \theta/2$, so

$$\mathrm{E}(k\bar{X}_n) = k\,\mathrm{E}(\bar{X}) = k\,\mathrm{E}(X) = k\theta/2 .$$

Hence, $k\bar{X}$ is unbiased for θ if $k = 2$. Note also that $\sigma^2 = \mathrm{Var}(X_i) = \theta^2/12$, e.g. using properties given in the notes on common distributions. Thus,

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}(2\bar{X}_n) = 4\sigma^2/n = 4\theta^2/(12n) = \theta^2/(3n) .$$

As $n \to \infty$, $\mathrm{Var}(\hat{\theta}) \to 0$.

10. $X_1, \dots, X_n$ are a random sample from $\mathrm{Bi}(m, p)$, and

$$\hat{p} = \frac{\bar{X}_n + 1}{n + 2} .$$

By linearity of expectations, we have that

$$\mathrm{E}(\hat{p}) = \frac{\mathrm{E}(\bar{X}_n) + 1}{n + 2} = \frac{mp + 1}{n + 2} .$$

Hence the bias is given by

$$\begin{aligned} \mathrm{bias}(\hat{p}) = \mathrm{E}(\hat{p}) - p &= \frac{mp + 1}{n + 2} - p \\ &= \frac{mp + 1 - np - 2p}{n + 2} \\ &= \frac{1 - p(n + 2 - m)}{n + 2} \end{aligned}$$

As $n \to \infty$, $\mathrm{bias}(\hat{p}) \to -p$, because $\frac{1}{n+2} \to 0$ and $\frac{n+2-m}{n+2} \to 1$ as $n \to \infty$ (note that m is fixed in value). The variance is

$$\begin{aligned} \mathrm{Var}(\hat{p}) &= \frac{1}{(n+2)^2}\,\mathrm{Var}(\bar{X}_n) \\ &= \frac{mp(1-p)}{n(n+2)^2} \to 0 \qquad \text{as } n \to \infty . \end{aligned}$$

Hence for large n, the estimator $\hat{p}$ is in fact very highly concentrated around the wrong value. Thus we would not recommend the use of this estimator.