Simple Expressions for VaR & ES

If X is an absolutely continuous RV then

$$VaR_p(X) = F_X^{-1}(p)$$

and

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

where $F_X^{-1}(\cdot)$ denotes the inverse CDF of X.

Ex 1 Suppose $X \sim N(\mu, \sigma^2)$. Find $VaR_p(X)$ and $ES_p(X)$. The CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

where $\Phi(\cdot)$ denotes the CDF of $N(0,1)$.

Set

$$F_X(x) = p$$

$$\Rightarrow \Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{x-\mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow x = \mu + \sigma\Phi^{-1}(p)$$

$$\Rightarrow \boxed{VaR_p(X) = \mu + \sigma\Phi^{-1}(p).}$$

$$\Rightarrow ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p \left[\mu + \sigma\Phi^{-1}(t)\right] dt$$

$$= \frac{1}{p}\left[\mu p + \int_0^p \sigma\Phi^{-1}(t)\,dt\right]$$

$$= \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt$$

$$\Rightarrow \boxed{ES_p(X) = \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\,dt}$$

Ex 2 Suppose $X \sim \text{Uniform}[a, b]$.

The CDF of X is

$$F_X(x) = \frac{x-a}{b-a}.$$

Set $F_X(x) = p$

$$\Rightarrow \frac{x-a}{b-a} = p$$

$$\Rightarrow x = a + p(b-a)$$

$$\Rightarrow \boxed{VaR_p(X) = a + p(b-a)}$$

$$\Rightarrow ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= \frac{1}{p}\int_0^p \left[a + t(b-a)\right] dt$$

$$= \frac{1}{p}\left[at + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= \frac{1}{p}\left[ap + \frac{p^2}{2}(b-a)\right]$$

$$\Rightarrow \boxed{ES_p(X) = a + \frac{p}{2}(b-a)}$$

Ex 3 Suppose $X \sim Exp(\lambda)$. The CDF of X is

$$F_X(x) = 1 - e^{-\lambda x}.$$

Set $F_X(x) = p$

$$\Rightarrow 1 - e^{-\lambda x} = p$$

$$\Rightarrow e^{-\lambda x} = 1 - p$$

$$\Rightarrow -\lambda x = \log(1-p)$$

$$\Rightarrow x = -\frac{1}{\lambda}\log(1-p)$$

$$\Rightarrow \boxed{VaR_p(X) = -\frac{1}{\lambda}\log(1-p)}$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= -\frac{1}{\lambda p}\int_0^p \log(1-t)\,dt$$

$$= -\frac{1}{\lambda p}\left\{\left[t\cdot\log(1-t)\right]_0^p - \int_0^p t\cdot\frac{(-1)}{1-t}\,dt\right\}$$

$\boxed{\text{Integration by parts}}$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) - 0 - \int_0^p \frac{(1-t)-1}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) - \int_0^p \left(1-\frac{1}{1-t}\right)dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) - \left[t+\log(1-t)\right]_0^p\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) - p - \log(1-p)\right\}.$$

$$\boxed{ES_p(X) = -\frac{1}{\lambda p}\left\{p\cdot\log(1-p) - p - \log(1-p)\right\}}$$