C) Semi-parametric estimation methods for ES

1) Method 1

Suppose $x_1, x_2, \ldots$ are losses. Then $r_t = x_t - x_{t-1}$ are the returns.

Suppose $\{r_t\}$ is a heavy tailed process if

$$P(r_t > -x) \sim x^{-\alpha} L(x)$$

as $x \longrightarrow \infty$, where $\alpha > 0$ and

$$\frac{L(tx)}{L(t)} \longrightarrow 1$$

as $t \longrightarrow \infty$

"$\sim$" $\equiv$ "behaves as"

If $\{r_t\}$ are heavy tailed then

$$\widehat{ES}_p = \frac{1}{p}\int_0^p \exp\left[\left(\frac{l_{n,p}}{nq}\right)^{\frac{1}{\alpha_{l_{n,p},n}}} \cdot r_{l_{n,p}}\right] dq - 1$$

where

$$l_{n,p} = [n(p+0.05)],$$

$$\alpha_{l,n} = \left[\frac{1}{l}\sum_{i=1}^{l}\log\left(\frac{r_{(i)}}{r_{(l)}}\right)\right]^{-1}$$

and

$r_{(1)} \leq r_{(2)} \leq \cdots \leq r_{(n)}$ are the returns arranged in increasing order.

2) Method 2

Under the notation of Method 1,

$$\widehat{ES}_p = \frac{1}{p}\int_{\frac{k}{n}}^{p} \hat{F}^{-1}(t)\,dt + \frac{k\, r_{(n-k)}}{np(1-\hat{\gamma})},$$

where $\hat{F}(\cdot)$ denotes the empirical CDF of $\{r_t\}$,

$$\hat{\gamma} = \frac{1}{k}\sum_{i=1}^{k} \log \frac{r_{(n-i+1)}}{r_{(n-k)}}$$

and $1 \leq k \leq n$.