Representations for possible forms for G

1) If F_X and F_Y belong to the Gumbel limit then

$$G(x, y) = \exp\left[-\int_0^1 \min\left[\frac{f_1(s)}{e^x}, \frac{f_2(s)}{e^y}\right] ds\right]$$

where $f_1(s)$ and $f_2(s)$ are non-negative functions satisfying

$$\int_0^1 f_1(s)\, ds = 1$$

and

$$\int_0^1 f_2(s)\, ds = 1.$$

2) If F_X and F_Y belong to the Gumbel limit then

$$G(x, y) = \exp\left[-\left(e^{-x} + e^{-y}\right) k(y-x)\right]$$

where $k(\cdot)$ satisfies

$$\lim_{t \to \infty} k(t) = 1,$$

$$\lim_{t \to -\infty} k(t) = 1,$$

$$\frac{d}{dt}\left[\left(1+e^{-t}\right) k(t)\right] \leq 0 \quad \forall t,$$

$$\frac{d}{dt}\left[\left(1+e^{t}\right) k(t)\right] \geq 0 \quad \forall t,$$

$$\left(1+e^{-t}\right) \frac{d^2 k(t)}{dt^2} + \left(1-e^{-t}\right) \frac{dk(t)}{dt} \geq 0 \; \forall t.$$

3) If F_X and F_Y belong to the Fréchet limit then

$$G(x, y) = \exp\left[-\left(\frac{1}{x}+\frac{1}{y}\right) A\left(\frac{x}{x+y}\right)\right]$$

where $A : [0, 1] \rightarrow [0, 1]$ satisfies

$$A(0) = 1,$$
$$A(1) = 1,$$
$$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall w$$

and $A(\cdot)$ is convex.

A function $f(\cdot)$ is convex if and only if

$$\frac{d^2 f(x)}{dx^2} > 0 \quad \forall x$$

4) Suppose

$$G_X(x) = 1 - e^{-x}, \quad x > 0$$

and $$G_Y(y) = 1 - e^{-y}, \quad y > 0.$$

Exponential marginals

Then

$$\bar{G}(x, y) = \exp\left[-(x+y)\,A\left(\frac{y}{x+y}\right)\right]$$

where $A : [0, 1] \to [0, 1]$ satisfies

$$A(0) = 1,$$

$$A(1) = 1,$$

$$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall w$$

and $A(\cdot)$ is convex.

5) If F_X and F_Y belong to the Weibull limit then no representation is known for G. This is an open problem for research.

Ex 1

Show that

$$\bar{G}(x, y) = \exp\left[-\frac{\theta y^2}{x+y} + \theta y - x - y\right],$$

$$x > 0, y > 0$$

corresponds to a bivariate extreme value distribution.

We see

$$G_X(x) = 1 - \bar{G}(x, 0) = 1 - e^{-x}$$

$$G_Y(y) = 1 - \bar{G}(0, y) = 1 - e^{-y},$$

We need to show that $\bar{G}$ satisfies representation 4.

We can write

$$\bar{G}(x, y) = \exp\left[-(x+y)\left[\frac{\theta y^2}{(x+y)^2} - \frac{\theta y}{x+y} + 1\right]\right].$$

$$\Rightarrow \quad A\left(\frac{y}{x+y}\right) = \frac{\theta y^2}{(x+y)^2} - \frac{\theta y}{x+y} + 1.$$

$$\Rightarrow \quad A(w) = \theta w^2 - \theta w + 1$$

(i) $A(0) = \theta \cdot 0^2 - \theta \cdot 0 + 1 = 1$ ✓

(ii) $A(1) = \theta \cdot 1^2 - \theta \cdot 1 + 1 = 1$ ✓

(iii) We need to show that

$\max(w, 1-w) \leq A(w) \leq 1$

$\Leftrightarrow$ a) $A(w) \leq 1 \quad \forall w$

b) $A(w) \geq w \quad \forall w$

c) $A(w) \geq 1-w \quad \forall w$

a) $A(w) \leq 1 \quad \forall w$

$\Leftrightarrow \theta w^2 - \theta w + 1 \leq 1 \quad \forall w$

$\Leftrightarrow \theta w^2 - \theta w \leq 0 \quad \forall w$

$\Leftrightarrow \theta w(w-1) \leq 0 \quad \forall w$ ✓

b) $A(w) \geq w \quad \forall w$

$\Leftrightarrow \theta w^2 - \theta w + 1 \geq w \quad \forall w$

$\Leftrightarrow \theta w^2 - \theta w + 1 - w \geq 0 \quad \forall w$

$\Leftrightarrow \theta w(w-1) + 1 - w \geq 0 \quad \forall w$

$\Leftrightarrow (1-w)(1-\theta w) \geq 0 \quad \forall w$ ✓

since $\underline{0 \leq \theta \leq 1}$

c) $A(w) \geq 1 - w \ \forall w$

$\Leftrightarrow \quad \theta w^2 - \theta w + 1 \geq 1 - w \ \forall w$

$\Leftrightarrow \quad \theta w^2 - \theta w + w \geq 0 \ \forall w$

$\Leftrightarrow \quad \theta w^2 + (1-\theta) w \geq 0 \ \forall w$ ✓

since $\underline{0 \leq \theta \leq 1}$

Hence, we have shown

$\max(w, 1-w) \leq A(w) \leq 1 \ \forall w.$

(iv) $A(w) = \theta w^2 - \theta w + 1$

$\Rightarrow \dfrac{dA(w)}{dw} = 2\theta w - \theta$

$\Rightarrow \dfrac{d^2 A(w)}{dw^2} = 2\theta > 0$

Hence $A(\cdot)$ is convex.

Hence, $\bar{G}$ corresponds to a bivariate extreme value distribution.