Proof of the properties of VaR

1) normalised property

$$\rho(0) = 0$$
$$\Leftrightarrow \quad VaR_p(0) = 0$$

which always holds

2) translative property

$$\rho(X+c) = \rho(X) + c$$
$$\Leftrightarrow \quad VaR_p(X+c) = VaR_p(X) + c$$
$$\Leftrightarrow \quad F_{X+c}^{-1}(p) = F_X^{-1}(p) + c$$
$$\Leftrightarrow \quad F_{X+c}\left(F_{X+c}^{-1}(p)\right) = F_{X+c}\left(F_X^{-1}(p)+c\right)$$
$$\Leftrightarrow \quad p = F_{X+c}\left(F_X^{-1}(p) + c\right)$$
$$\Leftrightarrow \quad p = P\left(X+\cancel{c} \leq F_X^{-1}(p)+\cancel{c}\right)$$
$$\Leftrightarrow \quad p = P\left(X \leq F_X^{-1}(p)\right)$$
$$\Leftrightarrow \quad p = F_X\left(F_X^{-1}(p)\right)$$
$$\Leftrightarrow \quad p = p$$

Hence the property holds

3) <u>monotone property</u>

$$X \leq Y$$

$$\Rightarrow F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \forall p$$

$$\Rightarrow VaR_p(X) \leq VaR_p(Y) \quad \forall p$$

Hence the property holds

4) <u>positive homogeneity property</u>

$$\rho(cX) = c \cdot \rho(X)$$

$$\Leftrightarrow VaR_p(cX) = c \cdot VaR_p(X)$$

$$\Leftrightarrow F_{cX}^{-1}(p) = c \cdot F_X^{-1}(p)$$

$$\Leftrightarrow F_{cX}(F_{cX}^{-1}(p)) = F_{cX}(c \cdot F_X^{-1}(p))$$

$$\Leftrightarrow p = F_{cX}(c \cdot F_X^{-1}(p))$$

$$\Leftrightarrow p = P(\cancel{c}X \leq \cancel{c} \cdot F_X^{-1}(p))$$

$$\Leftrightarrow p = P(X \leq F_X^{-1}(p))$$

$$\Leftrightarrow p = F_X(F_X^{-1}(p))$$

$$\Leftrightarrow p = p$$

Hence the property holds.

5) <u>sub-additivity property</u>

For a proof that this property does not hold for VaR please see the course website.

Proof of the properties for ES

1) normalised property

We know $VaR_p(0) = 0$

$$\Rightarrow \quad ES_p(0) = \frac{1}{p}\int_0^p VaR_t(0)\,dt$$

$$= \frac{1}{p}\int_0^p 0 \cdot dt$$

$$= 0$$

Hence the property holds.

2) translative property

We know $VaR_p(X+c) = VaR_p(X) + c$

$$\Rightarrow \quad ES_p(X+c) = \frac{1}{p}\int_0^p VaR_t(X+c)\,dt$$

$$= \frac{1}{p}\int_0^p \left[VaR_t(X) + c \right] dt$$

$$= \frac{1}{p}\int_0^p VaR_t(X)\,dt + c$$

$$= ES_p(X) + c.$$

Hence, the property holds.

3) monotone property

We know $X \leq Y \Rightarrow VaR_p(X) \leq VaR_p(Y)$

$$\Rightarrow \int_0^p VaR_t(X)\,dt \leq \int_0^p VaR_t(Y)\,dt$$

$$\Rightarrow \frac{1}{p}\int_0^p VaR_t(X)\,dt \leq \frac{1}{p}\int_0^p VaR_t(Y)\,dt$$

$$\Rightarrow ES_p(X) \leq ES_p(Y).$$

Hence, the property holds.

4) positive homogeneity property

We know $VaR_p(cX) = c \cdot VaR_p(X)$

$$\Rightarrow \int_0^p VaR_t(cX)\,dt = c \cdot \int_0^p VaR_t(X)\,dt$$

$$\Rightarrow \frac{1}{p}\int_0^p VaR_t(cX)\,dt = c \cdot \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$\Rightarrow ES_p(cX) = c \cdot ES_p(X)$$

Hence, the property holds

5) sub-additivity property

For a proof, please see the course web site.