Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are IID with joint CDF F. Let

$$(M_{n,1}, M_{n,2}) = \left(\max(X_1, \ldots, X_n), \max(Y_1, \ldots, Y_n)\right).$$

If there exist $a_n > 0$, $b_n \in \mathbb{R}$, $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$P\left(\frac{M_{n,1} - b_n}{a_n} \leq x, \frac{M_{n,2} - d_n}{c_n} \leq y\right)$$

$$= \left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

$$\rightarrow G(x, y)$$

as $n \rightarrow \infty$ for a non-degenerate CDF G then possible forms for G can be uncountably infinite.

If G exists it is known the <u>bivariate</u> <u>extreme</u> <u>value</u> <u>distribution</u> (BEVD)

How to determine G if it exists

The procedure is

(i) Find the marginal CDFs F_X and F_Y of F.

(ii) Determine the max domains of attraction of F_X and F_Y

(iii) a) If F_X and F_Y belong to the Gumbel limit then take

$$a_n = \gamma_X \left(F_X^{-1}\left(1-\frac{1}{n}\right)\right),$$

$$b_n = F_X^{-1}\left(1-\frac{1}{n}\right),$$

$$c_n = \gamma_Y\left(F_Y^{-1}\left(1-\frac{1}{n}\right)\right),$$

$$d_n = F_Y^{-1}\left(1-\frac{1}{n}\right)$$

b) If F_X and F_Y belong to the Fréchet limit then take

$$a_n = F_X^{-1}\left(1-\frac{1}{n}\right),$$

$$b_n = 0,$$

$$c_n = F_Y^{-1}\left(1-\frac{1}{n}\right),$$

$$d_n = 0$$

c) If F_X and F_Y belong to the Weibull limit then take

$$a_n = \omega(F_X) - F_X^{-1}\left(1-\frac{1}{n}\right),$$

$$b_n = \omega(F_X),$$

$$c_n = \omega(F_Y) - F_Y^{-1}\left(1-\frac{1}{n}\right),$$

$$d_n = \omega(F_Y).$$

(iv) Determine G as

$$G(x, y) = \lim_{n \to \infty} \left[F(a_n x + b_n, c_n y + d_n)\right]^n$$

Ex 1 Suppose

$$F_{X,Y}(x,y) = \left[1+e^{-x}+e^{-y}+(1-a)e^{-x-y}\right]^{-1}.$$

Find G if it exists.

(i) $$F_X(x) = F_{X,Y}(x,\infty)$$
$$= \left[1+e^{-x}\right]^{-1}$$

and

$$F_Y(y) = F_{X,Y}(\infty,y)$$
$$= \left[1+e^{-y}\right]^{-1}.$$

(ii) Set $F_X(x) = 1$

$$\Rightarrow \left[1+e^{-x}\right]^{-1} = 1$$
$$\Rightarrow 1+e^{-x} = 1$$
$$\Rightarrow e^{-x} = 0$$
$$\Rightarrow -x = -\infty$$
$$\Rightarrow x = +\infty$$
$$\Rightarrow w(F_X) = +\infty$$

$$I: \lim_{t\to\infty} \frac{1-F_X(t+x\gamma_X(t))}{1-F_X(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \left[1 + e^{-t - x\gamma_X(t)}\right]^{-1}}{1 - \left[1 + e^{-t}\right]^{-1}}$$

$$= \lim_{t \to \infty} \frac{\dfrac{e^{-t-x\gamma_X(t)}}{1 + e^{-t-x\gamma_X(t)}}}{\dfrac{e^{-t}}{1 + e^{-t}}}$$

$e^{-t-x\gamma_X(t)} \to 0$, $e^{-t} \to 0$

$$= \lim_{t \to \infty} \frac{e^{-t-x\gamma_X(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma_X(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma_X(t) \equiv 1 \quad \forall t$$

Hence, condition I holds & F_X belongs to the Gumbel limit.

Similarly, F_Y also belongs to the Gumbel limit with $\gamma_Y(t) \equiv 1 \ \forall t$.

(iii)

$$a_n = \gamma_X\left(F_X^{-1}\left(1-\frac{1}{n}\right)\right) = 1,$$

Set $F_X(x) = 1 - \frac{1}{n}$

$$\Rightarrow [1+e^{-x}]^{-1} = 1-\frac{1}{n}$$

$$\Rightarrow 1+e^{-x} = \frac{n}{n-1}$$

$$\Rightarrow e^{-x} = \frac{n}{n-1} - 1 = \frac{1}{n-1}$$

$$\Rightarrow -x = -\log(n-1)$$

$$\Rightarrow x = \log(n-1)$$

$$\Rightarrow F_X^{-1}\left(1-\frac{1}{n}\right) = \log(n-1)$$

$$\Rightarrow b_n = \log(n-1).$$

Similarly, $c_n = 1$ & $d_n = \log(n-1)$.

(iv)

$$G(x,y) = \lim_{n \to \infty} \left[F(a_n x + b_n, c_n y + d_n) \right]^n$$

$$= \lim_{n \to \infty} \Big[1 + e^{-a_n x - b_n} + e^{-c_n y - d_n}$$

$$+ (1-a)\, e^{-a_n x - b_n - c_n y - d_n} \Big]^{-n}$$

$$= \lim_{n \to \infty} \Big[1 + e^{-x - \log(n-1)} + e^{-y - \log(n-1)}$$

$$+ (1-a)\, e^{-x - \log(n-1) - y - \log(n-1)} \Big]^{-n}$$

$$= \lim_{n \to \infty} \left[1 + \frac{e^{-x}}{n-1} + \frac{e^{-y}}{n-1} + (1-a) \frac{e^{-x-y}}{(n-1)^2} \right]^{-n}$$

$$= \lim_{n \to \infty} \Big[1 + \frac{e^{-x}}{n} \cdot \frac{n}{n-1} + \frac{e^{-y}}{n} \cdot \frac{n}{n-1}$$

$$+ \frac{(1-a)\, e^{-x-y}}{n} \cdot \frac{n}{(n-1)^2} \Big]^{-n}$$

$$= \lim_{n \to \infty} \Big[1 + \frac{1}{n} \underbrace{\Big[e^{-x} \cdot \frac{n}{n-1} + e^{-y} \cdot \frac{n}{n-1} + (1-a)\, e^{-x-y} \cdot \frac{n}{(n-1)^2} \Big]}_{= Z} \Big]^{-n}$$

$$= \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^{-n}$$

$$= \left[\lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^{n}\right]^{-1}$$

$$= \lim_{n \to \infty} \left[e^{z}\right]^{-1}$$

$$= \lim_{n \to \infty} e^{-z}$$

$$= \lim_{n \to \infty} e^{-\left[e^{-x} \cdot \underbrace{\frac{n}{n-1}}_{\to 1} + e^{-y} \cdot \underbrace{\frac{n}{n-1}}_{\to 1} + (1-a)e^{-x-y} \cdot \underbrace{\frac{n}{(n-1)^2}}_{\to 0}\right]}$$

$$= e^{-\left[e^{-x} + e^{-y}\right]} .$$

Hence, $G(x, y) = e^{-e^{-x} - e^{-y}}$.