Preliminaries

A) Bivariate case

Suppose (X_1, X_2) has joint CDF F_{X_1, X_2}. This means

$$F_{X_1, X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2).$$

The marginal CDFs are

$$F_{X_1}(x_1) = F_{X_1, X_2}(x_1, \infty) = P(X_1 \leq x_1)$$

and

$$F_{X_2}(x_2) = F_{X_1, X_2}(\infty, x_2) = P(X_2 \leq x_2).$$

The joint survival function of (X_1, X_2) is

$$\bar{F}_{X_1, X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2).$$

The marginal survival functions are

$$\bar{F}_{X_1}(x_1) = P(X_1 > x_1) = \bar{F}_{X_1, X_2}(x_1, -\infty)$$

and

$$\bar{F}_{X_2}(x_2) = P(X_2 > x_2) = \bar{F}_{X_1, X_2}(-\infty, x_2).$$

The marginal CDFs can be obtained from $\bar{F}$ as

$$F_{X_1}(x_1) = P(X_1 \leq x_1) = 1 - P(X_1 > x_1)$$
$$= 1 - \bar{F}_{X_1, X_2}(x_1, -\infty)$$

and

$$F_{X_2}(x_2) = P(X_2 \leq x_2) = 1 - P(X_2 > x_2)$$
$$= 1 - \bar{F}_{X_1, X_2}(-\infty, x_2).$$

There are relationships between F_{X_1, X_2} and $\bar{F}_{X_1, X_2}$:

$$\bar{F}_{X_1, X_2}(x_1, x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2)$$
$$+ F_{X_1, X_2}(x_1, x_2)$$

and

$$F_{X_1, X_2}(x_1, x_2) = 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2)$$
$$+ \bar{F}_{X_1, X_2}(x_1, x_2).$$

The joint PDF of (X_1, X_2) is

$$f_{X_1, X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1, X_2}(x_1, x_2)$$
$$= \frac{\partial^2}{\partial x_1 \partial x_2} \bar{F}_{X_1, X_2}(x_1, x_2)$$

B) Trivariate case

Suppose (X_1, X_2, X_3) has join CDF

$$F_{X_1, X_2, X_3}(x_1, x_2, x_3)$$

$$= P(X_1 \leq x_1, X_2 \leq x_2, X_3 \leq x_3).$$

The marginal CDFs are

$$F_{X_1}(x_1) = F_{X_1, X_2, X_3}(x_1, \infty, \infty),$$

$$F_{X_2}(x_2) = F_{X_1, X_2, X_3}(\infty, x_2, \infty),$$

$$F_{X_3}(x_3) = F_{X_1, X_2, X_3}(\infty, \infty, x_3).$$

The joint survival function of (X_1, X_2, X_3) is

$$\bar{F}_{X_1, X_2, X_3}(x_1, x_2, x_3)$$

$$= P(X_1 > x_1, X_2 > x_2, X_3 > x_3).$$

The marginal survival functions of X_1, X_2 and X_3 are

$$\bar{F}_{X_1}(x_1) = \bar{F}_{X_1, X_2, X_3}(x_1, -\infty, -\infty),$$

$$\bar{F}_{X_2}(x_2) = \bar{F}_{X_1, X_2, X_3}(-\infty, x_2, -\infty),$$

$$\bar{F}_{X_3}(x_3) = \bar{F}_{X_1, X_2, X_3}(-\infty, -\infty, x_3).$$

The relationships between F_{X_1, X_2, X_3} and $\bar{F}_{X_1, X_2, X_3}$ are

$$\bar{F}_{X_1, X_2, X_3}(x_1, x_2, x_3)$$
$$= 1 - F_{X_1}(x_1) - F_{X_2}(x_2) - F_{X_3}(x_3)$$
$$+ F_{X_1, X_2}(x_1, x_2) + F_{X_1, X_3}(x_1, x_3)$$
$$+ F_{X_2, X_3}(x_2, x_3) - F_{X_1, X_2, X_3}(x_1, x_2, x_3)$$

and

$$F_{X_1, X_2, X_3}(x_1, x_2, x_3)$$
$$= 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) - \bar{F}_{X_3}(x_3)$$
$$+ \bar{F}_{X_1, X_2}(x_1, x_2) + \bar{F}_{X_1, X_3}(x_1, x_3) + \bar{F}_{X_2, X_3}(x_2, x_3)$$
$$- \bar{F}_{X_1, X_2, X_3}(x_1, x_2, x_3)$$

The joint PDF of (X_1, X_2, X_3) is

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3)$$
$$= \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} F_{X_1, X_2, X_3}(x_1, x_2, x_3)$$

$$= - \frac{\partial^2}{\partial x_1 \partial x_2 \partial x_3} \bar{F}_{X_1, X_2, X_3}(x_1, x_2, x_3)$$

c) <u>p- variate</u> <u>case</u>

Suppose $(X_1, \ldots, X_p)$ has joint CDF

$$F_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$
$$= P(X_1 \leq x_1, \ldots, X_p \leq x_p).$$

The marginal CDFs are

$$F_{X_1}(x_1) = F_{X_1, \ldots, X_p}(x_1, \infty, \ldots, \infty)$$
$$\vdots$$
$$F_{X_p}(x_p) = F_{X_1, \ldots, X_p}(\infty, \infty, \ldots, x_p)$$

The joint survival function of $(X_1, \ldots, X_p)$ is

$$\bar{F}_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$
$$= P(X_1 > x_1, \ldots, X_p > x_p).$$

The marginal survival functions are

$$\bar{F}_{X_1}(x_1) = \bar{F}_{X_1, \ldots, X_p}(x_1, -\infty, \ldots, -\infty)$$
$$\vdots$$
$$\overline{F_{X_p}}(x_2) = \bar{F}_{X_1, \ldots, X_p}(-\infty, -\infty, \ldots, x_p).$$

The relationships between $F_{X_1, \ldots, X_p}$ and $\bar{F}_{X_1, \ldots, X_p}$ are

$$\bar{F}_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

$$= 1 - \sum_{i=1}^{p} F_{X_i}(x_i) + \sum_{i<j} F_{X_i, X_j}(x_i, x_j) -$$

$$+ \cdots + (-1)^p F_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

and

$$F_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

$$= 1 - \sum_{i=1}^{p} \bar{F}_{X_i}(x_i) + \sum_{i<j} \bar{F}_{X_i, X_j}(x_i, x_j) -$$

$$+ \cdots + (-1)^p \bar{F}_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

The joint PDF of $(X_1, \ldots, X_p)$ is

$$f_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

$$= \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

$$= (-1)^p \frac{\partial^p}{\partial x_1 \cdots \partial x_p} \bar{F}_{X_1, \ldots, X_p}(x_1, \ldots, x_p)$$

Ex 10 Suppose (X_1, X_2) has the joint survival function

$$\bar{F}_{X_1, X_2}(x_1, x_2) = \frac{1}{1 + x_1 + x_2}$$

for $x_1 > 0$ and $x_2 > 0$. Find the following

(i) $\bar{F}_{X_1}(x_1)$

(ii) $\bar{F}_{X_2}(x_2)$

(iii) $F_{X_1}(x_1)$

(iv) $F_{X_2}(x_2)$

(v) $F_{X_1, X_2}(x_1, x_2)$

(vi) $f_{X_1, X_2}(x_1, x_2)$

(i) $\bar{F}_{X_1}(x_1) = \bar{F}_{X_1, X_2}(x_1, 0) = \frac{1}{1+x_1}$

(ii) $\bar{F}_{X_2}(x_2) = \bar{F}_{X_1, X_2}(0, x_2) = \frac{1}{1+x_2}$

(iii) $F_{X_1}(x_1) = 1 - \bar{F}_{X_1}(x_1) = 1 - \frac{1}{1+x_1} = \frac{x_1}{1+x_1}$

(iv) $F_{X_2}(x_2) = 1 - \bar{F}_{X_2}(x_2) = 1 - \frac{1}{1+x_2} = \frac{x_2}{1+x_2}$

(v) $F_{X_1, X_2}(x_1, x_2)$

$$= 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) + \bar{F}_{X_1, X_2}(x_1, x_2)$$

$$= 1 - \frac{1}{1+x_1} - \frac{1}{1+x_2} + \frac{1}{1+x_1+x_2}$$

(vi) $f_{X_1, X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2}\left[\frac{1}{1+x_1+x_2}\right]$

$$= \frac{\partial}{\partial x_1}\left[-\frac{1}{(1+x_1+x_2)^2}\right]$$

$$= \frac{2}{(1+x_1+x_2)^3}.$$

Ex 11 Suppose (X_1, X_2) has the joint survival function

$$\bar{F}_{X_1, X_2}(x_1, x_2) = e^{-x_1 - x_2 - \theta x_1 x_2}$$

for $x_1 > 0$ and $x_2 > 0$. Find the following

(i) $\bar{F}_{X_1}(x_1)$

(ii) $\bar{F}_{X_2}(x_2)$

(iii) $F_{X_1}(x_1)$

(iv) $F_{X_2}(x_2)$

(v) $F_{X_1, X_2}(x_1, x_2)$

(vi) $f_{X_1, X_2}(x_1, x_2)$

(i) $\bar{F}_{X_1}(x_1) = \bar{F}_{X_1, X_2}(x_1, 0) = e^{-x_1}$

(ii) $\bar{F}_{X_2}(x_2) = \bar{F}_{X_1, X_2}(0, x_2) = e^{-x_2}$

(iii) $F_{X_1}(x_1) = 1 - \bar{F}_{X_1}(x_1) = 1 - e^{-x_1}$

(iv) $F_{X_2}(x_2) = 1 - \bar{F}_{X_2}(x_2) = 1 - e^{-x_2}$

(v) $$F_{X_1, X_2}(x_1, x_2) = 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) + \bar{F}_{X_1, X_2}(x_1, x_2)$$

$$= 1 - e^{-x_1} - e^{-x_2} + e^{-x_1 - x_2 - \theta x_1 x_2}$$

(vi) $$f_{X_1, X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2}\left[e^{-x_1 - x_2 - \theta x_1 x_2}\right]$$

$$= \frac{\partial}{\partial x_1}\left[-(1 + \theta x_1) e^{-x_1 - x_2 - \theta x_1 x_2}\right]$$

$$= (1 + \theta x_1)(1 + \theta x_2) e^{-x_1 - x_2 - \theta x_1 x_2} - \theta e^{-x_1 - x_2 - \theta x_1 x_2}.$$