# Estimation of VaR

A) Parametric estimation methods

B) Non-parametric " "

C) Semi-parametric " "

# A) Parametric estimation methods for VaR

Suppose $\hat{\theta}$ is an estimator of $\theta$ based on a random sample of size $n$. The $\hat{\theta}$ is $\boxed{\text{Unbiased}}$ if

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta = 0.$$

$\hat{\theta}$ is $\boxed{\text{asymptotically}}$ $\boxed{\text{Unbiased}}$ if

$$\lim_{n \to \infty} \text{Bias}(\hat{\theta}) = 0.$$

$\hat{\theta}$ is $\boxed{\text{consistent}}$ if

$$\lim_{n \to \infty} \text{MSE}(\hat{\theta}) = 0$$

where $\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$

$\boxed{\text{Mean Squared Error}}$

From Math 20812 Statistical Methods

My material for Math20812 are in

https://minerva.it.manchester.ac.uk/~saralees/MATH20802.html

1) Normal distribution

Suppose $X_1, \ldots, X_n$ are IID $N(\mu, \sigma^2)$.

We know

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p).$$

The MLEs of $\mu$ and $\sigma$ are

$$\hat{\mu} = \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

From Math20802

Hence, the MLE of VaR is

$$\widehat{VaR}_p(X) = \hat{\mu} + \hat{\sigma} \Phi^{-1}(p).$$

It can be shown that $\widehat{VaR}_p(X)$ is asymptotically unbiased & consistent.

## 2) Uniform distribution

Suppose $X_1, \ldots, X_n$ are IID $Uni[a, b]$.
We know

$$VaR_p(X) = a + p(b-a).$$

$$\boxed{\begin{array}{l} \text{The MLEs of } a \text{ and } b \text{ are} \\ \hat{a} = \min(X_1, \ldots, X_n) \\ \text{and} \\ \hat{b} = \max(X_1, \ldots, X_n) \end{array}}$$

From Math20802

Hence, the MLE of VaR is

$$\widehat{VaR_p(X)} = \hat{a} + p(\hat{b} - \hat{a}).$$

It can be shown that $\widehat{VaR_p(X)}$ is asymptotically unbiased & consistent.

## 3) Exponential distribution

Suppose $X_1, \ldots, X_n$ are IID $\mathrm{Exp}(\lambda)$.

We know

$$\mathrm{VaR}_p(X) = -\frac{1}{\lambda} \log(1-p)$$

The MLE of $\lambda$ is

$$\boxed{\hat{\lambda} = \frac{1}{\bar{x}}}$$

From Math 20802

Hence, the MLE of VaR is

$$\widehat{\mathrm{VaR}}_p(X) = -\bar{x} \log(1-p).$$

It can be shown that $\widehat{\mathrm{VaR}}_p(X)$ is unbiased & consistent.

## 4) Power function distribution

Suppose $X_1, \ldots, X_n$ are IID with CDF $F(x) = x^a$, $0 < x < 1$. We set

$$F(x) = p$$

$$\Leftrightarrow \quad x^a = p$$

$$\Leftrightarrow \quad x = p^{\frac{1}{a}}$$

$$\Leftrightarrow \quad VaR_p(X) = p^{\frac{1}{a}}.$$

The likelihood function of $a$ is

$$L(a) = \prod_{i=1}^{n} \left[ a X_i^{a-1} \right]$$

$$= a^n \left( \prod_{i=1}^{n} X_i \right)^{a-1}.$$

The log likelihood function is

$$\log L(a) = n \log a + (a-1) \sum_{i=1}^{n} \log X_i .$$

The derivative is

$$\frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log X_i = 0$$

$$\Rightarrow \quad \hat{a} = - \frac{n}{\sum_{i=1}^{n} \log X_i}.$$

The 2nd order derivative is

$$\frac{d^2 \log L(a)}{da^2} = - \frac{n}{a^2} < 0.$$

Hence, $\hat{a} = -\dfrac{n}{\sum_{i=1}^{n} \log X_i}$ is an MLE of a.

Hence, the MLE of VaR is

$$\widehat{VaR}_p(X) = p^{\frac{1}{\hat{a}}}$$

$$= p^{-\frac{1}{n}\sum_{i=1}^{n} \log X_i}$$

5. Variance - covariance method

Suppose a portfolio has $m$ independent investments. Let

$X_i$ = Loss on investment $i$

$w_i$ = weight on investment $i$.

Then the total portfolio loss is

$$X = \sum_{i=1}^{m} w_i X_i .$$

Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ independently. Then

$$X \sim N\left(\sum_{i=1}^{m} w_i \mu_i , \sum_{i=1}^{m} w_i^2 \sigma_i^2\right)$$

Set

$$F_X(x) = p$$

$$\Leftrightarrow \Phi\left(\frac{x - \sum_{i=1}^{m} w_i \mu_i}{\sqrt{\sum_{i=1}^{m} w_i^2 \sigma_i^2}}\right) = p$$

$$\Leftrightarrow x = \sum_{i=1}^{m} w_i \mu_i + \sqrt{\sum_{i=1}^{m} w_i^2 \sigma_i^2}\, \Phi^{-1}(p)$$

$$\Leftrightarrow VaR_p(X) = \sum_{i=1}^{m} w_i \mu_i + \sqrt{\sum_{i=1}^{m} w_i^2 \sigma_i^2}\, \Phi^{-1}(p)$$

Suppose the following data:

$$X_{1,1}, X_{1,2}, \ldots, X_{1,n_1} \quad \text{IID on } X_1$$

$$X_{2,1}, X_{2,2}, \ldots, X_{2,n_2} \quad \text{IID on } X_2$$

$$\vdots$$

$$X_{m,1}, X_{m,2}, \ldots, X_{m,n_m} \quad \text{IID on } X_m$$

Then the MLEs of $\mu_i$ and $\sigma_i^2$ are

$$\widehat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{i,j}$$

and

$$\widehat{\sigma_i^2} = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \left( X_{i,j} - \widehat{\mu}_i \right)^2$$

From Math20802

Hence, the MLE of $\underline{VaR_p(X)}$ is

$$\widehat{VaR}_p(X) = \sum_{i=1}^{m} w_i \widehat{\mu}_i + \sqrt{\sum_{i=1}^{m} w_i^2 \widehat{\sigma_i^2}}\, \Phi^{-1}(p)$$

## 6. Weibull distribution

Suppose $X_1, \ldots, X_n$ are IID with CDF

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\beta}, \quad x > 0.$$

Set

$$F(x) = p$$

$$\Leftrightarrow \quad 1 - e^{-\left(\frac{x}{\theta}\right)^\beta} = p$$

$$\Leftrightarrow \quad e^{-\left(\frac{x}{\theta}\right)^\beta} = 1 - p$$

$$\Leftrightarrow \quad \left(\frac{x}{\theta}\right)^\beta = -\log(1-p)$$

$$\Leftrightarrow \quad x = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$\Leftrightarrow \quad VaR_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

It can be shown that $\hat{\theta}$ and $\hat{\beta}$ are the solutions of

$$\frac{\bar{x}^2}{s^2} = \frac{\left[\Gamma\left(1+\frac{1}{\hat{\beta}}\right)\right]^2}{\Gamma\left(1+\frac{2}{\hat{\beta}}\right) - \left[\Gamma\left(1+\frac{1}{\hat{\beta}}\right)\right]^2}$$

and

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Hence, an estimator VaR is

$$\widehat{VaR}_p(X) = \hat{\theta}\left[-\log(1-p)\right]^{\frac{1}{\hat{\beta}}}.$$