Estimation methods for ES

A) Parametric estimation methods

B) Nonparametric " "

C) Semi-parametric " "

A) Parametric estimation methods for ES

1) Normal distribution

Suppose $x_1, \ldots, x_n$ are IID $N(\mu, \sigma^2)$.
We know

$$ES_p(X) = \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t)\, dt.$$

We also know that the MLEs of μ and σ are

$$\hat{\mu} = \bar{x}$$

and

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

Hence, the MLE of $ES_p(X)$ is

$$\widehat{ES}_p(X) = \hat{\mu} + \frac{\hat{\sigma}}{p} \int_0^p \Phi^{-1}(t)\, dt.$$

It can be shown that $\widehat{ES}_p(X)$ is asymptotically unbiased & consistent.

2) <u>Uniform distribution</u>

Suppose $x_1, \ldots, x_n$ are IID Uni$[a, b]$.

We know

$$ES_p(X) = a + \frac{p}{2}(b-a).$$

We also know that the MLEs of a & b are

$$\hat{a} = \min(X_1, \ldots, X_n)$$

and

$$\hat{b} = \max(X_1, \ldots, X_n).$$

Hence, the MLE of $ES_p(X)$ is

$$\widehat{ES_p(X)} = \hat{a} + \frac{p}{2}(\hat{b} - \hat{a}).$$

It can be shown that $\widehat{ES_p(X)}$ is asymptotically unbiased & consistent.

3) Power function distribution

Suppose $x_1, \ldots, x_n$ are IID with CDF $F(x) = x^a$, $0 < x < 1$. We know

$$\mathrm{VaR}_p(X) = p^{\frac{1}{a}}.$$

So,

$$
\begin{aligned}
ES_p(X) &= \frac{1}{p} \int_0^p \mathrm{VaR}_t(X)\, dt \\
&= \frac{1}{p} \int_0^p t^{\frac{1}{a}}\, dt \\
&= \frac{1}{p} \left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1} \right]_0^p \\
&= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}.
\end{aligned}
$$

We also know that the MLE of a is

$$\widehat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}.$$

Hence, the MLE of $ES_p(X)$ is

$$\widehat{ES}_p(X) = \frac{p^{\frac{1}{\widehat{a}}}}{\frac{1}{\widehat{a}}+1}.$$

4) Weibull distribution

Suppose $x_1, \ldots, x_n$ are IID with CDF $F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\beta}$. We know

$$\mathrm{VaR}_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}.$$

So,

$$ES_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_t(X)\,dt$$

$$= \frac{\theta}{p}\int_0^p \left[-\log(1-t)\right]^{\frac{1}{\beta}}\,dt$$

$$\boxed{\begin{aligned} &\text{Set } y = -\log(1-t) \\ &\Rightarrow 1-t = e^{-y} \\ &\Rightarrow t = 1-e^{-y} \\ &\Rightarrow \frac{dt}{dy} = e^{-y} \end{aligned}}$$

$$= \frac{\theta}{p}\int_0^{-\log(1-p)} y^{\frac{1}{\beta}} e^{-y}\,dy$$

$$= \frac{\theta}{p}\,\gamma\!\left(\frac{1}{\beta}+1, -\log(1-p)\right)$$

where

$$\boxed{\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t}\,dt}$$

is the upper incomplete gamma function.

The estimators of θ and β are the solutions of

$$\frac{\bar{x}^2}{s^2} = \frac{\left[\Gamma\left(1+\frac{1}{\hat{\beta}}\right)\right]^2}{\Gamma\left(1+\frac{2}{\hat{\beta}}\right) - \left[\Gamma\left(1+\frac{1}{\hat{\beta}}\right)\right]^2}$$

and

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)},$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Hence, the estimator of $ES_p(X)$ is

$$\widehat{ES}_p(X) = \frac{\hat{\theta}}{p}\gamma\left(\frac{1}{\hat{\beta}}+1, -\log(1-p)\right).$$