MATH11712 Statistics I
Semester 2, 2022 / 2023
Example Sheet 10

Please submit your solutions to question 3 through blackboard by 12noon Monday 24 April.

1. Let $X_1, \ldots, X_n$ be a random sample of from $\text{Bi}(1, p)$, where the value of p is unknown. The approximate $100(1-\alpha)\%$ confidence intervals for p discussed in lectures were of the form:

$$\left[\,\hat{p} - z_{1-\alpha/2}\,\widehat{\text{s.e.}}(\hat{p})\,,\,\hat{p} + z_{1-\alpha/2}\,\widehat{\text{s.e.}}(\hat{p})\,\right],$$

where $\widehat{\text{s.e.}}(\hat{p})$ denotes an estimate of the standard error of $\hat{p}$,

$$\text{s.e.}(\hat{p}) = \sqrt{p(1-p)/n}\,.$$

(i) Show that for fixed α, the above confidence interval has width proportional to $\widehat{\text{s.e.}}(\hat{p})$.

(ii) Sketch the graph of the function $v(p) = p(1-p)$ for $p \in (0, 1)$.

(iii) Confirm algebraically that $v(p)$ is maximized when $p = 0.5$, and state its maximum value.

(iv) Hence find an upper bound for the width of the 95% CI when $n = 1000$.

2. A random sample of 150 light bulbs from Brand A had a mean lifetime of 1386 hours and a standard deviation of 114 hours. A random sample of 200 light bulbs from Brand B had a mean lifetime of 1218 hours and a standard deviation of 98 hours. Find an approximate 95% CI for the difference in the mean lifetimes of Brand A and Brand B bulbs assuming that

(i) $\sigma_A^2 \neq \sigma_B^2$;

(ii) $\sigma_A^2 = \sigma_B^2$,

where σ_A^2 and σ_B^2 denote the true variances of the lifetimes of brand A and brand B bulbs, respectively. Report your conclusions regarding the difference in the mean lifetimes for the two brands.

3. $*$ Consider a distribution with mean μ and variance σ^2, both unknown. From this distribution, two random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are obtained. The second sample is independent of the first. The sample means are $\bar{X}$ and $\bar{Y}$ respectively, and the sample variances are S_X^2 and S_Y^2.

It is proposed to use the following estimators of μ and σ^2:

$$\hat{\mu} = \frac{n\bar{X}_n + m\bar{Y}_m}{n+m}\,,$$

$$\hat{\sigma}^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}\,.$$

(i) $*$ Show that $\hat{\mu}$ and $\hat{\sigma}^2$ are both unbiased. [2 marks]

(ii) $*$ Obtain an expression for $\text{Var}(\hat{\mu})$. [2 marks]

(iii) $*$ What happens to $\text{Var}(\hat{\mu})$ as both $n \to \infty$ and $m \to \infty$? [1 mark]

4. Two competing headache remedies claim to give fast acting relief. An experiment was performed to compare the mean lengths of time required for bodily absorption of headache remedies from Brand A and Brand B.

Two random samples of 12 patients were obtained. Those in the first sample received a dose of Brand A, while those in the second sample received a dose of Brand B.

For each patient, the response recorded was the time (in minutes) for the drug to reach a specified concentration in the bloodstream, giving the following results:

Brand A: $n_1 = 12$, $\bar{x}_1 = 21.8$, $s_1 = 8.7$

Brand B: $n_2 = 12$, $\bar{x}_2 = 18.9$, $s_2 = 7.5$

Assuming that the absorption times for the two brands are normally distributed with a common, but unknown value of σ^2, find a 95% confidence interval for $\mu_1 - \mu_2$, the mean difference in absorption times between Brands A and B. Report your conclusions.

5. A random sample of $n_1 = 288$ voters registered in the US state of California, of which 141 voted in the last presidential election. From a random sample of $n_2 = 216$ registered voters in the US state of Colorado, 125 voted in the last presidential election.

Construct an approximate 95% confidence interval for the difference in the proportions voting in California and Colorado at the last presidential election.

Can you conclude whether the population proportions are the same or different?