MATH10282 Introduction to Statistics
Semester 2, 2020/2021
Example Sheet 11

1. The manufacturer of a certain type of cable claims that the mean breaking strength is 200kg with a standard deviation of 30kg. Assume that the breaking strengths are normally distributed, and that the claimed standard deviation is correct. Consider testing the following pairs of hypotheses on the basis of the observed breaking strengths of a random sample of $n = 40$ lengths of cable.

(i) $H_0 : \mu = 200$ vs $H_1 : \mu \neq 200$;

(ii) $H_0 : \mu = 200$ vs $H_1 : \mu > 200$.

In each of the two cases, determine the rejection region for a test at significance level $\alpha = 0.05$ and at significance level $\alpha = 0.01$.

Carry out the tests in (i) and (ii) (using both given values of α) when a sample of $n = 40$ data values is obtained with $\sum_{i=1}^{40} x_i = 8328.4$. Report your conclusions.

2. Assume that the data $x_1, \ldots, x_{10}$ were generated by random sampling from a normal distribution with variance $\sigma^2 = 3$ but unknown mean μ. The sample mean is $\bar{x} = 0.7$.

(i) Use these data to test $H_0 : \mu = 0$ vs $H_1 : \mu > 0$ at the 5% significance level.

(ii) Calculate the probability of a type II error if the true value of $\mu = 1$ and also if the true value of $\mu = 1.5$. Comment on the results.

(iii) Assuming the sample mean remained the same, how large would the sample size need to be in order for H_0 to be rejected in favour of H_1?

3. A manufacturer claims that the light bulbs they produce have a mean life of 1800 hours. A random sample of $n = 250$ bulbs is tested. The lifetimes are found to have a sample mean of 1794.6 hours and a sample variance of 2484 h^2. Test $H_0 : \mu = 1800$ vs $H_1 : \mu \neq 1800$ at the approximate 5% significance level. Compute the value of your test statistic and report your conclusions.

4. A manufacturer claims that the lifetimes of the batteries it produces are normally distributed with a mean of at least 250 hours and a standard deviation of 8.7 hours. A random sample of $n = 20$ batteries are found to have a sample mean lifetime of 252.96 hours. In order to examine the manufacturer's claim, we wish to test $H_0 : \mu = 250$ vs $H_1 : \mu > 250$.

(i) Use an appropriate test statistic to carry out the test at the 5% significance level. Report your conclusions.

(ii) Calculate the probability that your test will correctly reject H_0 when the true value of the mean is 255.0 hours and also when the true mean is 257 hours. Comment on the results.

5. The marks of all the candidates in an English examination may be taken to be normal with mean 64, but the standard deviation σ is unknown. A random sample of $n = 54$ candidates from a particular school obtained a mean mark of 66.3 with a sample standard deviation of $s = 9.2$. Does this suggest that this school differs in English performance from the norm? To assess this, test $H_0 : \mu = 64$ vs $H_1 : \mu \neq 64$ using a significance level of 5%. Report your conclusions.

6. A noted US journalist claims that over half the US population is concerned with the lack of educational programmes shown on television. To gauge opinion, a national polling organization collected data for $n = 920$ randomly chosen individuals. Of those surveyed, 478 said that they are concerned about the lack of educational TV programmes.

Let p denote the true, but unknown, proportion of the whole population who are concerned about this issue.

(i) Propose a suitable test statistic for testing $H_0 : p = 0.5$ vs $H_1 : p > 0.5$.

(ii) Using a normal approximation to the distribution of your test statistic, state an appropriate critical value for this test that ensures an approximate significance level of 5%.

(iii) Carry out the test and report your conclusions.

(iv) What is the approximate probability that the test procedure formulated in parts (i) and (ii) correctly rejects the above H_0 when the true value of p is 0.55?

7. We are told that the proportion of left-handed people in the population is 10%. A random sample of 400 people is obtained, of which 47 are found to be are left-handed. Let p denote the proportion of left-handed people in the population.

(i) Test $H_0 : p = 0.1$ vs $H_1 : p \neq 0.1$ at the approximate 5% significance level, Do the results of the test contradict what we have been told?

(ii) Calculate the approximate probability that we correctly reject H_0 when the true value of p is 0.103 and also when $p = 0.105$.

Q (i)

$$\boxed{\begin{array}{l} H_0 : \mu = \mu_0 \\ \text{vs } H_1 : \mu \neq \mu_0 \\ \text{Reject } H_0 \text{ if } \left| \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \right| > z_{1-\frac{\alpha}{2}} \end{array}}$$

$\mu_0 = 200$

$n = 40$

$\sigma = 30$

$\bar{x} = \frac{1}{40} \times 8328.4 = 208.21$

$$\left| \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \right| = \left| \frac{208.21 - 200}{30/\sqrt{40}} \right| = 1.731$$

If $\alpha = 0.05$ then $z_{1-\frac{\alpha}{2}} = z_{0.975} = 1.96$

There is no evidence against H_0

If $\alpha = 0.01$ then $z_{1-\frac{\alpha}{2}} = z_{0.995} = 2.576$

Still there is no evidence against H_0.

Q1 (ii)

$$\boxed{\begin{array}{l} H_0 : \mu = \mu_0 \\ \text{vs } H_1 : \mu > \mu_0 \\ \text{Reject } H_0 \text{ if } \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha} \end{array}}$$

$$\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{208.21 - 200}{30/\sqrt{40}} = 1.731$$

If $\alpha = 0.05$ then $z_{1-\alpha} = z_{0.95} = 1.645$.
There is evidence against H_0.
If $\alpha = 0.01$ then $z_{1-\alpha} = z_{0.99} = 2.326$.
There is no evidence against H_0.

Q2 (i) We reject H_0 if

$$\frac{\bar{x} - 0}{\sqrt{3}/\sqrt{10}} > z_{1-0.05}$$

$$\Leftrightarrow \frac{0.7}{\sqrt{0.3}} > z_{0.95}$$

$$\Leftrightarrow 1.275 > 1.645 \text{ is not true}$$

There is no evidence against H_0.

(ii) $P(\text{Type II error})$

$$= P(\text{Accept } H_0 \mid H_1 \text{ true})$$

$$= P\left(\frac{\bar{x}}{\sqrt{0.3}} \leq z_{0.95} \,\middle|\, \mu = 1\right)$$

$$= P\left(\frac{(\bar{x}-1)+1}{\sqrt{0.3}} \leq z_{0.95} \,\middle|\, \mu = 1\right)$$

$$= P\left(\frac{\bar{x}-1}{\sqrt{0.3}} \leq z_{0.95} - \frac{1}{\sqrt{0.3}} \,\middle|\, \mu = 1\right)$$

If $X_1, \ldots, X_n$ IID $N(\mu, \sigma^2)$ then

$$\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

$$= P\left(N(0,1) \leq z_{0.95} - \frac{1}{\sqrt{0.3}}\right)$$

$$= \Phi\left(1.645 - \frac{1}{\sqrt{0.3}}\right)$$

$$= \cancel{0.14}\ 0.43$$

(ii) $P(\text{Type II error})$

$= P(\text{Accept } H_0 \mid H_1 \text{ true})$

$= P\left(\frac{\bar{x}}{\sqrt{0.3}} \leq z_{0.95} \,\middle|\, \mu = 1.5\right)$

$= P\left(\frac{(\bar{x} - 1.5) + 1.5}{\sqrt{0.3}} \leq z_{0.95} \,\middle|\, \mu = 1.5\right)$

$= P\left(\frac{\bar{x} - 1.5}{\sqrt{0.3}} \leq z_{0.95} - \frac{1.5}{\sqrt{0.3}} \,\middle|\, \mu = 1.5\right)$

$= P\left(N(0,1) \leq z_{0.95} - \frac{1.5}{\sqrt{0.3}}\right)$

$= \Phi\left(1.645 - \frac{1.5}{\sqrt{0.3}}\right)$

$= 0.14$

(iii)

$$\frac{\bar{x}}{\sqrt{3}/\sqrt{n}} > z_{0.95} = 1.645$$

$$\Leftrightarrow \quad \frac{0.7\sqrt{n}}{\sqrt{3}} > 1.645$$

$$\Leftrightarrow \quad \sqrt{n} > \frac{\sqrt{3}}{0.7} \times 1.645$$

$$\Leftrightarrow \quad n > \frac{3}{(0.7)^2} \times (1.645)^2 = 16.5675$$

$$\Leftrightarrow \quad n \geq 17.$$

Q3 $n = 250$

$H_0 : \mu = 1800$

vs $H_1 : \mu \neq 1800$

Reject H_0 if $\left| \frac{\bar{x} - 1800}{s/\sqrt{n}} \right| > t_{1-\frac{\alpha}{2}}(n-1)$

Since n is large, we can say that an approximate test is to reject H_0 if

$$\left| \frac{\bar{x} - 1800}{s/\sqrt{n}} \right| > z_{1-\frac{\alpha}{2}}$$

$$LHS = \left| \frac{1794.6 - 1800}{\sqrt{2484}/\sqrt{250}} \right| = 1.713$$

$$RHS = z_{0.975} = 1.96$$

Since LHS < RHS there is no evidence to reject H_0.

Q4 (i) $H_0 : \mu = 250$

vs $H_1 : \mu > 250$

We reject H_0 if

$$\frac{\bar{x} - 250}{\sigma/\sqrt{n}} > z_{1-\alpha}$$

$$\Leftrightarrow \quad \frac{252.96 - 250}{8.7/\sqrt{20}} > z_{0.95}$$

$$\Leftrightarrow \quad 1.521 > 1.645 \quad \text{not true}$$

There is no evidence to reject H_0

(ii) $P(\text{Reject } H_0 \mid \mu = 255)$

$$= P\left(\frac{\bar{x} - 250}{8.7/\sqrt{20}} > 1.645 \,\middle|\, \mu = 255 \right)$$

$$= P\left(\frac{\bar{x} - 255 + 5}{8.7/\sqrt{20}} > 1.645 \,\middle|\, \mu = 255 \right)$$

$$= P\left(\frac{\bar{x} - 255}{8.7/\sqrt{20}} > 1.645 - \frac{5\sqrt{20}}{8.7} \,\middle|\, \mu = 255 \right)$$

If $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$ then

$$\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

$$= P\left(N(0,1) > 1.645 - \frac{5\sqrt{20}}{8.7} \right)$$

$$= 1 - P\left(N(0,1) \leq 1.645 - \frac{5\sqrt{20}}{8.7}\right)$$

$$= 1 - \Phi\left(1.645 - \frac{5\sqrt{20}}{8.7}\right)$$

$$= 0.82$$

$$P(\text{Reject } H_0 \mid \mu = 257)$$

$$= P\left(\frac{\bar{x} - 250}{8.7/\sqrt{20}} > 1.645 \,\middle|\, \mu = 257\right)$$

$$= P\left(\frac{\bar{x} - 257 + 7}{8.7/\sqrt{20}} > 1.645 \,\middle|\, \mu = 257\right)$$

$$= P\left(\frac{\bar{x} - 257}{8.7/\sqrt{20}} > 1.645 - \frac{7}{8.7/\sqrt{20}} \,\middle|\, \mu = 257\right)$$

$$= P\left(N(0,1) > 1.645 - \frac{7\sqrt{20}}{8.7}\right)$$

$$= 1 - P\left(N(0,1) \leq 1.645 - \frac{7\sqrt{20}}{8.7}\right)$$

$$= 1 - \Phi\left(1.645 - \frac{7\sqrt{20}}{8.7}\right)$$

$$= 0.98$$

Q5

$H_0 : \mu = 64$

$H_1 : \mu \neq 64$

Reject H_0 if

$$\frac{\bar{x} - 64}{s/\sqrt{n}} > t_{1-\frac{\alpha}{2}}(n-1)$$

$$\Leftrightarrow \quad \frac{66.3 - 64}{9.2/\sqrt{54}} > t_{0.975}(53)$$

$$\Leftrightarrow \quad 1.837 > 2.006 \quad \text{not true}$$

There is no evidence to reject H_0.

Q6 (i) $H_0 : p = 0.5$

vs $H_1 : p > 0.5$

Test statistic is

$$\frac{\bar{x} - 0.5}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}}$$

where $\bar{x}$ = sample proportion.

(ii) Reject H_0 if

$$\frac{\bar{x} - 0.5}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > z_{1-\alpha}$$

(iii) $$LHS = \frac{\frac{478}{920} - 0.5}{\sqrt{\frac{\frac{478}{920}\left(1 - \frac{478}{920}\right)}{920}}} = 1.1869$$

$$RHS = z_{1-0.05} = z_{0.95} = 1.645$$

There is no evidence against H_0.

(iv) $P(\text{Reject } H_0 \mid p = 0.55)$

$$= P\left(\frac{\bar{x} - 0.5}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > 1.645 \,\middle|\, p = 0.55 \right)$$

$$= P\left(\frac{\bar{x} - 0.55 + 0.05}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > 1.645 \,\middle|\, p = 0.55 \right)$$

$$= P\left(\frac{\bar{x} - 0.55}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > 1.645 - \frac{0.05}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \,\middle|\, p = 0.55 \right)$$

If $X_1, \ldots, X_n$ IID $Bin(m, p)$ then

$$\bar{x} \overset{\cdot}{\sim} N\left(p, \frac{\bar{x}(1-\bar{x})}{n}\right)$$

$$\Rightarrow \frac{\bar{x} - p}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \overset{\cdot}{\sim} N(0, 1)$$

$$\approx P\left(N(0,1) > 1.645 - \frac{0.05}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right)$$

$$= 1 - P\left(N(0,1) \leq 1.645 - \frac{0.05}{\sqrt{\frac{\frac{478}{920}\left(1 - \frac{478}{920}\right)}{920}}} \right)$$

$$= 1 - \Phi\left(1.645 - \frac{0.05}{\sqrt{\frac{\frac{478}{920}\left(1 - \frac{478}{920}\right)}{920}}} \right)$$

$$= 0.92$$

Q7 (i) $H_0 : p = 0.1$

vs $H_1 : p \neq 0.1$

Reject H_0 if $\left| \dfrac{\bar{x} - 0.1}{\sqrt{\dfrac{\bar{x}(1-\bar{x})}{n}}} \right| > z_{1-\frac{\alpha}{2}}$

$$LHS = \left| \frac{\frac{47}{400} - 0.1}{\sqrt{\frac{\frac{47}{400}\left(1 - \frac{47}{400}\right)}{400}}} \right| = 1.166667$$

$$RHS = z_{1-0.025} = z_{0.975} = 1.96$$

There is no evidence against H_0.

Q7 (ii) $P(\text{Reject } H_0 \mid p = 0.103)$

$$= P\left(\left| \frac{\bar{x} - 0.1}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right| > 1.96 \;\middle|\; p = 0.103 \right)$$

$$= P\left(\left| \frac{\bar{x} - 0.103 + 0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right| > 1.96 \;\middle|\; p = 0.103 \right)$$

$$= P\left(\frac{\bar{x} - 0.103 + 0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > 1.96 \right.$$

$$\underline{\text{OR}} \quad \left. \frac{\bar{x} - 0.103 + 0.03}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} < -1.96 \;\middle|\; p = 0.103 \right)$$

$$\boxed{|z| > a \iff z > a \text{ OR } z < -a}$$

$$= P\left(\frac{\bar{x} - 0.103}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > 1.96 - \frac{0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \;\middle|\; p = 0.103 \right)$$

$$+ P\left(\frac{\bar{x} - 0.103}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} < -1.96 - \frac{0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \;\middle|\; p = 0.103 \right)$$

$$\approx P\left(N(0,1) > 1.96 - \frac{0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right)$$

$$+ P\left(N(0,1) < -1.96 - \frac{0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right)$$

$$= 1 - P\left(N(0,1) \leq 1.96 - \frac{0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}}\right)$$

$$+ P\left(N(0,1) < -1.96 - \frac{0.003}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}}\right)$$

$$= 1 - \Phi\left(1.96 - \frac{0.003}{\sqrt{\frac{\frac{47}{400}\left(1-\frac{47}{400}\right)}{400}}}\right)$$

$$+ \Phi\left(-1.96 - \frac{0.003}{\sqrt{\frac{\frac{47}{400}\left(1-\frac{47}{400}\right)}{400}}}\right)$$

$$= 0.058$$

Q7 (ii)

$$P(\text{Reject } H_0 \mid p = 0.105)$$

$$= P\left(\left| \frac{\bar{x} - 0.1}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right| > 1.96 \;\middle|\; p = 0.105 \right)$$

$$= P\left(\left| \frac{\bar{x} - 0.105 + 0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right| > 1.96 \;\middle|\; p = 0.105 \right)$$

$$= P\left(\frac{\bar{x} - 0.105 + 0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > 1.96 \right.$$

$$\underline{\text{OR}} \quad \left. \frac{\bar{x} - 0.105 + 0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} < -1.96 \;\middle|\; p = 0.105 \right)$$

$$= P\left(\frac{\bar{x} - 0.105}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} > 1.96 - \frac{0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \;\middle|\; p = 0.105 \right)$$

$$+ P\left(\frac{\bar{x} - 0.105}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} < -1.96 - \frac{0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \;\middle|\; p = 0.105 \right)$$

$$= P\left(N(0,1) > 1.96 - \frac{0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right)$$

$$+ P\left(N(0,1) < -1.96 - \frac{0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \right)$$

$$= 1 - P\left(N(0,1) \leq 1.96 - \frac{0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}}\right)$$

$$+ P\left(N(0,1) < -1.96 - \frac{0.005}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}}\right)$$

$$= 1 - \Phi\left(1.96 - \frac{0.005}{\sqrt{\frac{\frac{47}{400}\left(1-\frac{47}{400}\right)}{400}}}\right)$$

$$+ \Phi\left(-1.96 - \frac{0.005}{\sqrt{\frac{\frac{47}{400}\left(1-\frac{47}{400}\right)}{400}}}\right)$$

$$= 0.068.$$