Copula

Suppose (X, Y) is a random vector. There are many situations where X and Y will be dependent on each other.

Ex 1 $(X, Y) = $ (Oil price, Car price)

Ex2 $(X, Y) = $ (Oil price, Food price)

A copula can be used to model the dependence among 2 or more variables.

For 2 variables, a copula is a function

$$C : \underbrace{[0,1] \times [0,1]}_{\text{Unit square}} \longrightarrow \underbrace{[0,1]}_{\text{Unit interval}}$$

with

$$\left.\begin{array}{l} C(u, 1) = u \\ C(1, v) = v \end{array}\right\}$$

marginal CDFs of C are Uniform$[0,1]$

Suppose (X, Y) has joint CDF $F_{X,Y}(x, y)$ and marginal CDFs $F_X(x)$, $F_Y(y)$.

<u>Statement 1</u> For every $F_{X,Y}$ there is a corresponding copula.

<u>Proof</u>: We have

$$
\begin{aligned}
& F_{X,Y}(x, y) \\
&= P(X \leq x, Y \leq y) \\
&= P(F_X(X) \leq F_X(x), F_Y(Y) \leq F_Y(y)) \\
&= P(\mathrm{Uni}[0,1] \leq F_X(x), \mathrm{Uni}[0,1] \leq F_Y(y))
\end{aligned}
$$

Probability Integral Transform

$$
= C_1'(F_X(x), F_Y(y))
$$

The proof is complete.

Statement 2 For every C there is a corresponding $F_{X,Y}$.

Proof We have

$$
\begin{aligned}
& C(u, v) \\
&= P\left(\mathrm{Uni}[0,1] \leq u,\ \mathrm{Uni}[0,1] \leq v\right) \\
&= P\left(F_X^{-1}(\mathrm{Uni}[0,1]) \leq F_X^{-1}(u),\right. \\
&\qquad \left. F_Y^{-1}(\mathrm{Uni}[0,1]) \leq F_Y^{-1}(v)\right) \\
&= P\left(X \leq F_X^{-1}(u),\ Y \leq F_Y^{-1}(v)\right)
\end{aligned}
$$

Probability Integral Transform

$$
= F_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right).
$$

The proof is complete.

Two simple copulas

1) Independence copula is

$$C(u, v) = uv$$

$$\Leftrightarrow P(\text{Uni}[0,1] \leq u,\ \text{Uni}[0,1] \leq v) = uv$$

$$\Leftrightarrow P\left(F_X^{-1}(\text{Uni}[0,1]) \leq F_X^{-1}(u),\ F_Y^{-1}(\text{Uni}[0,1]) \leq F_Y^{-1}(v)\right) = uv$$

$$\Leftrightarrow P\Big(X \leq \overbrace{\boxed{F_X^{-1}(u)}}^{x},\ Y \leq \overbrace{\boxed{F_Y^{-1}(v)}}^{y}\Big) = uv$$

Probability Integral Transform

Set $x = F_X^{-1}(u) \Rightarrow u = F_X(x)$

" $y = F_Y^{-1}(v) \Rightarrow v = F_Y(y)$

$$\Leftrightarrow P(X \leq x,\ Y \leq y) = F_X(x)\, F_Y(y)$$

$$\Leftrightarrow \boxed{F_{X,Y}(x, y) = F_X(x)\, F_Y(y)}$$

usual definition of independence between X and Y.

2) Completely dependent copula is

$$C^1(u, v) = \min(u, v)$$

$$\Leftrightarrow P\left(\mathrm{Uni}[0,1] \le u, \mathrm{Uni}[0,1] \le v\right) = \min(u, v)$$

$$\Leftrightarrow P\left(F_X^{-1}(\mathrm{Uni}[0,1]) \le F_X^{-1}(u), F_Y^{-1}(\mathrm{Uni}[0,1]) \le F_Y^{-1}(v)\right) = \min(u, v)$$

$$\Leftrightarrow P\left(X \le F_X^{-1}(u), Y \le F_Y^{-1}(v)\right) = \min(u, v)$$

Probability Integral Transform

Set $x = F_X^{-1}(u) \Rightarrow u = F_X(x)$

" $y = F_Y^{-1}(v) \Rightarrow v = F_Y(y)$

$$\Leftrightarrow P(X \le x, Y \le y) = \min(F_X(x), F_Y(y))$$

$$\Leftrightarrow \boxed{F_{X,Y}(x, y) = \min(F_X(x), F_Y(y))}$$

usual definition of complete dependence between X and Y.