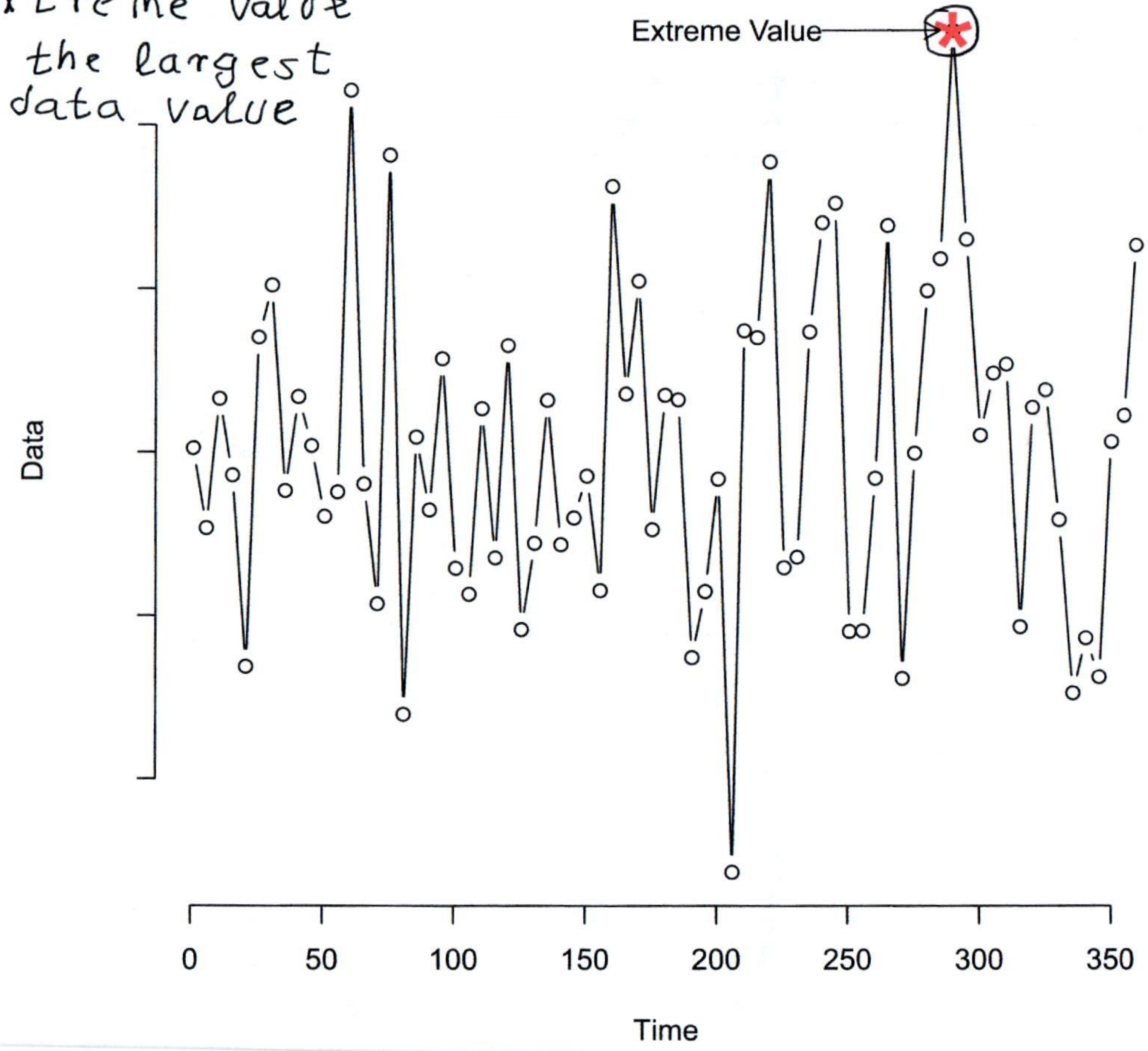
Definition 1

Extreme value = the largest data value



<u>Can the 3 limits be combined into 1.</u>

The ETT says that there can be 3 possible limits for the sample maximum. A practitioner may not know how to check the conditions I–III giving rise to the 3 limits. So it will be convenient if the 3 limits can be combined into one mathematical form.

The combined form is known as the <u>Generalised</u> <u>Extreme</u> <u>Value</u> (GEV) distribution.

GEV distribution

A random variable X follows the GEV distribution if its CDF is

$$G(x) = \exp\left[- \left(1 + \xi \frac{x-\mu}{\sigma} \right)^{-\frac{1}{\xi}} \right]$$

where $1 + \xi \frac{x-\mu}{\sigma} > 0$

$-\infty < \xi < \infty$ "shape parameter"

$\sigma > 0$ "scale parameter"

$-\infty < \mu < \infty$ "location parameter"

Notation : $X \sim GEV(\mu, \sigma, \xi)$

Domain of X

$\boxed{\xi < 0}$

$$1 + \xi \frac{x-\mu}{\sigma} > 0$$

$$\Leftrightarrow \quad \xi \frac{x-\mu}{\sigma} > -1$$

$$\Leftrightarrow \quad \frac{x-\mu}{\sigma} < -\frac{1}{\xi}$$

$$\Leftrightarrow \quad x < \mu - \frac{\sigma}{\xi}$$

domain is $\left(-\infty, \mu - \frac{\sigma}{\xi}\right)$

$\boxed{\xi = 0}$

$$1 + \xi \frac{x-\mu}{\sigma} = 1 > 0$$

$\Leftrightarrow$ domain is $(-\infty, \infty)$

$\boxed{\xi > 0}$

$$1 + \xi \frac{x-\mu}{\sigma} > 0$$

$$\Leftrightarrow \quad \xi \frac{x-\mu}{\sigma} > -1$$

$$\Leftrightarrow \quad \frac{x-\mu}{\sigma} > -\frac{1}{\xi}$$

$$\Leftrightarrow \quad x > \mu - \frac{\sigma}{\xi}$$

$\Leftrightarrow$ domain is $\left(\mu - \frac{\sigma}{\xi}, \infty\right)$

$$\text{Domain of } X = \begin{cases} \left(\mu - \frac{\sigma}{\xi}, \infty\right) & \text{if } \xi > 0 \\ (-\infty, \infty) & \text{if } \xi = 0 \\ \left(-\infty, \mu - \frac{\sigma}{\xi}\right) & \text{if } \xi < 0 \end{cases}$$

How come the GEV combines the 3 limiting distributions

$\boxed{\xi < 0}$

$$G(x) = \exp\left\{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\}$$

$$= \exp\left\{-\left(1-\frac{\xi\mu}{\sigma}+\frac{\xi x}{\sigma}\right)^{-\frac{1}{\xi}}\right\}$$

$$= \exp\left\{-\left(-\left(\underbrace{\boxed{\frac{\xi\mu}{\sigma}-1}}_{=b}+\underbrace{\boxed{\left(-\frac{\xi}{\sigma}\right)}}_{=a}x\right)\right)^{-\frac{1}{\xi}}\right\}$$

$\boxed{\begin{matrix} a > 0 \\ b \in \mathbb{R} \end{matrix}}$

$$= \exp\left\{-\left(-(ax+b)\right)^{\boxed{-\frac{1}{\xi}}=\alpha}\right\}$$

$\boxed{\alpha > 0}$

$$= \exp\left\{-\left(-(ax+b)\right)^{\alpha}\right\}$$

= same type as $\exp\left\{-(-x)^{\alpha}\right\}$

$\Longrightarrow$ Weibull limit is a particular case of the GEV for $\xi < 0$.

$\boxed{\xi > 0}$

$$G(x) = \exp\left\{ -\left(1 + \xi\,\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \right\}$$

$$= \exp\left\{ -\left(\underbrace{1 - \frac{\xi\mu}{\sigma}}_{=\,b} + \underbrace{\frac{\xi}{\sigma}}_{=\,a}\,x \right)^{-\underbrace{\frac{1}{\xi}}_{=\,\alpha}} \right\}$$

$$\boxed{\begin{array}{l} a > 0 \\ \alpha > 0 \\ b \in \mathbb{R} \end{array}}$$

$$= \exp\left\{ -(ax + b)^{-\alpha} \right\}$$

$=$ same type as $\exp(-x^{-\alpha})$

$\Longrightarrow$ Fréchet limit is a particular case of the GEV for $\xi > 0$.

$$\boxed{\xi = 0}$$

$$G(x) = \lim_{\xi \to 0} \exp\left\{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\}$$

$$= \lim_{\xi \to 0} \exp\left\{-\left(1+\frac{\frac{x-\mu}{\sigma}}{\frac{1}{\xi}}\right)^{-\frac{1}{\xi}}\right\}$$

$$\boxed{\text{Set} \quad m = \frac{1}{\xi}}$$

$$= \lim_{m \to \infty} \exp\left\{-\left(1+\frac{\frac{x-\mu}{\sigma}}{m}\right)^{-m}\right\}$$

$$= \lim_{m \to \infty} \exp\left\{-\left[\left(1+\frac{\frac{x-\mu}{\sigma}}{m}\right)^{m}\right]^{-1}\right\}$$

$$\boxed{\lim_{m \to \infty} \left(1+\frac{z}{m}\right)^{m} = e^{z}}$$

$$= \lim_{m \to \infty} \exp\left\{-\left[\exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-1}\right\}$$

$$= \exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}$$

$$= \exp\left\{-\exp\left(-ax+b\right)\right\}$$

$a = \frac{1}{\sigma} > 0, \quad b = \frac{\mu}{\sigma} \in \mathbb{R}$

= same type as $\exp\{-\exp(-x)\}$

$\Rightarrow$ Gumbel limit is the particular case of the GEV for $\xi = 0$.

- Gumbel limit is particular case for $\xi = 0$
- Weibull " " " " " $\xi < 0$
- Fréchet " " " " " $\xi > 0$

PDF of GEV distribution

$$g(x) = \frac{d}{dx} G(x)$$

$$= \frac{d}{dx} \exp\left[-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right]$$

$$= \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

$$\cdot \exp\left[-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right]$$

Quantile of GEV distribution

A quantile x_p of a RV X is defined by

$$P(X \leq x_p) = p \quad \ldots \quad (+)$$

For the GEV distribution,

$$(+) \iff G(x_p) = p$$

$$\iff \exp\left[-\left(1+\xi\frac{x_p-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right] = p$$

$$\iff -\left(1+\xi\frac{x_p-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = \log p$$

$$\iff 1+\xi\frac{x_p-\mu}{\sigma} = (-\log p)^{-\xi}$$

$$\iff \xi\frac{x_p-\mu}{\sigma} = (-\log p)^{-\xi} - 1$$

$$\iff \boxed{x_p = \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right]}$$

↑ pth quantile of the GEV distribution

In paricular

$$x_{\frac{1}{2}} = \text{Median of GEV}$$

$$= \mu + \frac{\sigma}{\xi}\left[(\log 2)^{-\xi} - 1\right]$$

Return level of GEV distribution

A return level with period T (in years) is the level expected to be exceeded on average once in every T years. Let x_T denote the return level with period T. Then

$$P(X > x_T) = \frac{1}{T}$$

$$\Leftrightarrow \quad P(X \leq x_T) = 1 - \frac{1}{T}$$

$$\Leftrightarrow \quad G(x_T) = 1 - \frac{1}{T}$$

$$\Leftrightarrow \quad \exp\left\{ -\left(1 + \xi \frac{x_T - \mu}{\sigma}\right)^{-\frac{1}{\xi}} \right\} = 1 - \frac{1}{T}$$

$$\Leftrightarrow \quad \boxed{x_T = \mu + \frac{\sigma}{\xi}\left[\left(-\log\left(1 - \frac{1}{T}\right)\right)^{-\xi} - 1\right]}$$

Estimation of GEV distribution

Suppose $x_1, x_2, \ldots, x_n$ are independent and identical data on $X \sim GEV(\mu, \sigma, \xi)$. The most popular method for estimation is the method of maximum likelihood.

The likelihood function of (μ, σ, ξ) is

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n} \left\{ \frac{1}{\sigma} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi} - 1} \cdot \exp\left[-\left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}} \right] \right\}$$

$$= \frac{1}{\sigma^n} \left[\prod_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right) \right]^{-\frac{1}{\xi} - 1} \cdot \exp\left[-\sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}} \right]$$

The log-likelihood function is

$$\log L(\mu, \sigma, \xi) = -n \log \sigma - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{n} \log\left(1 + \xi \frac{x_i - \mu}{\sigma}\right) - \sum_{i=1}^{n} \left(1 + \xi \frac{x_i - \mu}{\sigma}\right)^{-\frac{1}{\xi}}.$$

The MLEs of μ, σ and ξ are the simultaneous solutions

$$\frac{\partial \log L}{\partial \mu} = 0,$$

$$\frac{\partial \log L}{\partial \sigma} = 0,$$

$$\frac{\partial \log L}{\partial \xi} = 0.$$

MLE equations for the GEV distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= \frac{1+\xi}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0, \qquad \text{———} \quad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma}+\frac{1+\xi}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0, \qquad \text{———} \quad (2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= \frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-\mu}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-1} \\ &\quad -\frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-\mu}{\sigma}\right)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad +\frac{1}{\xi\sigma}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &= 0. \qquad \text{———} \quad (3)\end{aligned}$$

MLEs of μ, σ and ξ are the solutions of (1), (2) and (3).

The <u>fgev(·)</u> command in R can compute the MLEs of μ, σ and ξ.