A condition to check if ETT holds

Checking of conditions I–III can be time consuming. It is convenient to have a single condition to find out whether the ETT will hold or not.

Suppose X is a discrete RV with CDF $F(\cdot)$. Let $w(F)$ denote its upper end point. Then the ETT will hold if and only if

$$\boxed{\lim_{k \to w(F)} \frac{P(X=k)}{1-F(k-1)} = 0}$$

Example 1 Suppose $X_1, \ldots, X_n$ are IID (independent and identical) Geom(p). Does the ETT hold?

$$P(X = k) = p(1-p)^{k-1}, \; k = 1, 2, \ldots$$

$$w(F) = +\infty.$$

$$\lim_{k \to \infty} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{p(1-p)^{k-1}}{\cancel{1} - [\cancel{1} - (1-p)^{k-1}]}$$

$$= \lim_{k \to \infty} \frac{p(1-p)^{k-1}}{(1-p)^{k-1}}$$

$$= p \neq 0$$

Hence, the ETT fails to hold.

Example 2 Suppose $X_1, \ldots, X_n$ are IID $Bin(m, p)$. Does the ETT hold?

$$P(X = k) = \binom{m}{k} p^k (1-p)^{m-k},$$
$$k = 0, 1, \ldots, m$$

$$\Rightarrow \quad w(F) = m$$

$$\Rightarrow \quad \lim_{k \to m} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \frac{P(X = m)}{1 - F(m-1)}$$

$$= \frac{P(X = m)}{1 - P(X \leq m-1)}$$

$$= \frac{P(X = m)}{P(X > m-1)}$$

$$= \frac{P(X = m)}{P(X = m)}$$

$$= 1 \neq 0$$

Hence, the ETT fails to hold.

Example 3 Suppose $X_1, \ldots, X_n$ are IID with

$$P(X = k) = \frac{1}{N}, \quad k = 1, 2, \ldots, N$$

[discrete uniform]

Does the ETT hold?

$$\Rightarrow w(F) = N$$

$$\Rightarrow \lim_{k \to N} \frac{P(X = k)}{1 - F(k-1)}$$

$$= \frac{P(X = N)}{1 - F(N-1)}$$

$$= \frac{P(X = N)}{1 - P(X \leq N-1)}$$

$$= \frac{P(X = N)}{P(X > N-1)}$$

$$= \frac{P(X = N)}{P(X = N)}$$

$$= 1 \neq 0$$

Hence, the ETT fails to hold.