Distribution of extreme value under definition 1

Suppose $\overset{\text{data}}{\boxed{X_1, \dots, X_n}}$ are independently and identically distributed with $\boxed{\text{CDF}}$ $F(\cdot)$

Cumulative distribution function

By definition 1, the extreme value is

$$M_n = \max(X_1, \dots, X_n).$$

The CDF of M_n is

$$\begin{aligned}
&P(M_n \leq x) \\
&= P(\max(X_1, \dots, X_n) \leq x) \\
&= P(X_1 \leq x, \dots, X_n \leq x) \\
&\overset{\text{indep}}{=} P(X_1 \leq x) \cdots P(X_n \leq x) \\
&= F(x) \cdots F(x) \\
&= [F(x)]^n
\end{aligned}$$

People are usually interested in the behavior of M_n over large periods. That is, what is the distribution of M_n as $n \to \infty$?

$$\lim_{n \to \infty} P(M_n \leq x)$$

$$= \lim_{n \to \infty} [F(x)]^n$$

$$= \begin{cases} 0 & \text{if} \quad 0 \leq F(x) < 1 \\ 1 & \text{if} \quad F(x) = 1 \end{cases}$$

This is not a useful result for practical applications.

Suppose $X_1, \ldots, X_n$ are independent and identical with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ denote the sample mean.

1) $\lim_{n \to \infty} \bar{X} = \mu$ "Strong Law of Large Numbers"

Not a very useful result

scaling

2) $\lim_{n \to \infty} \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = N(0, 1)$ "Central limit theorem" (CLT)

CLT is a very useful result

We look at the limit of

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right).$$

$$\Rightarrow \lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right)$$

$$= \lim_{n\to\infty} P\left(M_n \leq b_n + a_n x\right)$$

$$= \lim_{n\to\infty} P\left(\max(X_1, \dots, X_n) \leq b_n + a_n x\right)$$

$$= \lim_{n\to\infty} P\left(X_1 \leq b_n + a_n x, \dots, X_n \leq b_n + a_n x\right)$$

$$\overset{\text{indep}}{=} \lim_{n\to\infty} P\left(X_1 \leq b_n + a_n x\right) \cdots P\left(X_n \leq b_n + a_n x\right)$$

$$= \lim_{n\to\infty} F(b_n + a_n x) \cdots F(b_n + a_n x)$$

$$= \lim_{n\to\infty} \left[F(b_n + a_n x)\right]^n \quad \dots \quad (*)$$

What is the limit of $(*)$?