Formal definition of a copula

A function $C : [0,1] \times [0,1] \rightarrow [0,1]$ is a copula if

(i) $C(u,0) = 0 \quad \forall u$

(ii) $C(0,v) = 0 \quad \forall v$

(iii) $C(u,1) = u \quad \forall u$

(iv) $C(1,v) = v \quad \forall v$

(v) $\frac{\partial}{\partial u} C(u,v) \geq 0 \quad \forall u,v$

(vi) $\frac{\partial}{\partial v} C(u,v) \geq 0 \quad \forall u,v$

Ex 1 Check $C(u, v) = uv$ is a copula.

(i) $C(u, 0) = u \cdot 0 = 0$ ✓

(ii) $C(0, v) = 0 \cdot v = 0$ ✓

(iii) $C(u, 1) = u \cdot 1 = u$ ✓

(iv) $C(1, v) = 1 \cdot v = v$ ✓

(v) $\frac{\partial}{\partial u} C(u, v) = v \geq 0$ ✓

(vi) $\frac{\partial}{\partial v} C(u, v) = u \geq 0$ ✓

Hence, C is a copula.

Ex 2 Check $C_1(u, v) = \min(u, v)$ is a copula.

(i) $C_1(u, 0) = \min(u, 0) = 0$ ✓

(ii) $C_1(0, v) = \min(0, v) = 0$ ✓

(iii) $C_1(u, 1) = \min(u, 1) = u$ ✓

(iv) $C_1(1, v) = \min(1, v) = v$ ✓

(v) $\frac{\partial}{\partial u} C_1(u, v) = \frac{\partial}{\partial u} \min(u, v)$

$$= \frac{\partial}{\partial u} \begin{cases} u & \text{if } u \leq v \\ v & \text{if } u > v \end{cases}$$

$$= \begin{cases} 1 & \text{if } u \leq v \\ 0 & \text{if } u > v \end{cases}$$

≥ 0 ✓

(vi) $\frac{\partial}{\partial v} C_1(u, v) = \frac{\partial}{\partial v} \min(u, v)$

$$= \frac{\partial}{\partial v} \begin{cases} u & \text{if } u \leq v \\ v & \text{if } u > v \end{cases}$$

$$= \begin{cases} 0 & \text{if } u \leq v \\ 1 & \text{if } u > v \end{cases}$$

≥ 0 ✓

Hence, C is a copula.

<u>Ex 3</u> Check $C'(u,v) = uv[1+\theta(1-u)(1-v)]$ is a copula, $-1 \le \theta \le 1$.

(i) $C'(u,0) = u \cdot 0 \cdot [1+\theta(1-u)] = 0$ ✓

(ii) $C'(0,v) = 0 \cdot v \cdot [1+\theta(1-v)] = 0$ ✓

(iii) $C'(u,1) = u \cdot 1 \cdot [1+\theta \cdot (1-u) \cdot 0] = u$ ✓

(v) $C'(1,v) = 1 \cdot v \cdot [1+\theta \cdot 0 \cdot (1-v)] = v$ ✓

(v) $\frac{\partial}{\partial u} C'(u,v) = v[1+\theta(1-u)(1-v)] + uv \cdot (-\theta) \cdot (1-v)$

$$= \underbrace{v}_{\geq 0}\Big[1+\underbrace{\underbrace{\theta}_{-1\to 1}\underbrace{(1-2u)}_{-1\to 1}\underbrace{(1-v)}_{0\to 1}}_{-1\to 1}\Big]$$

≥ 0 ✓

(vi) $\frac{\partial}{\partial v} C'(u,v) = \underbrace{u}_{\geq 0}\Big[1+\underbrace{\underbrace{\theta}_{-1\to 1}\underbrace{(1-2v)}_{-1\to 1}\underbrace{(1-u)}_{0\to 1}}_{-1\to 1}\Big]$

≥ 0 ✓

Hence, C' is a copula.