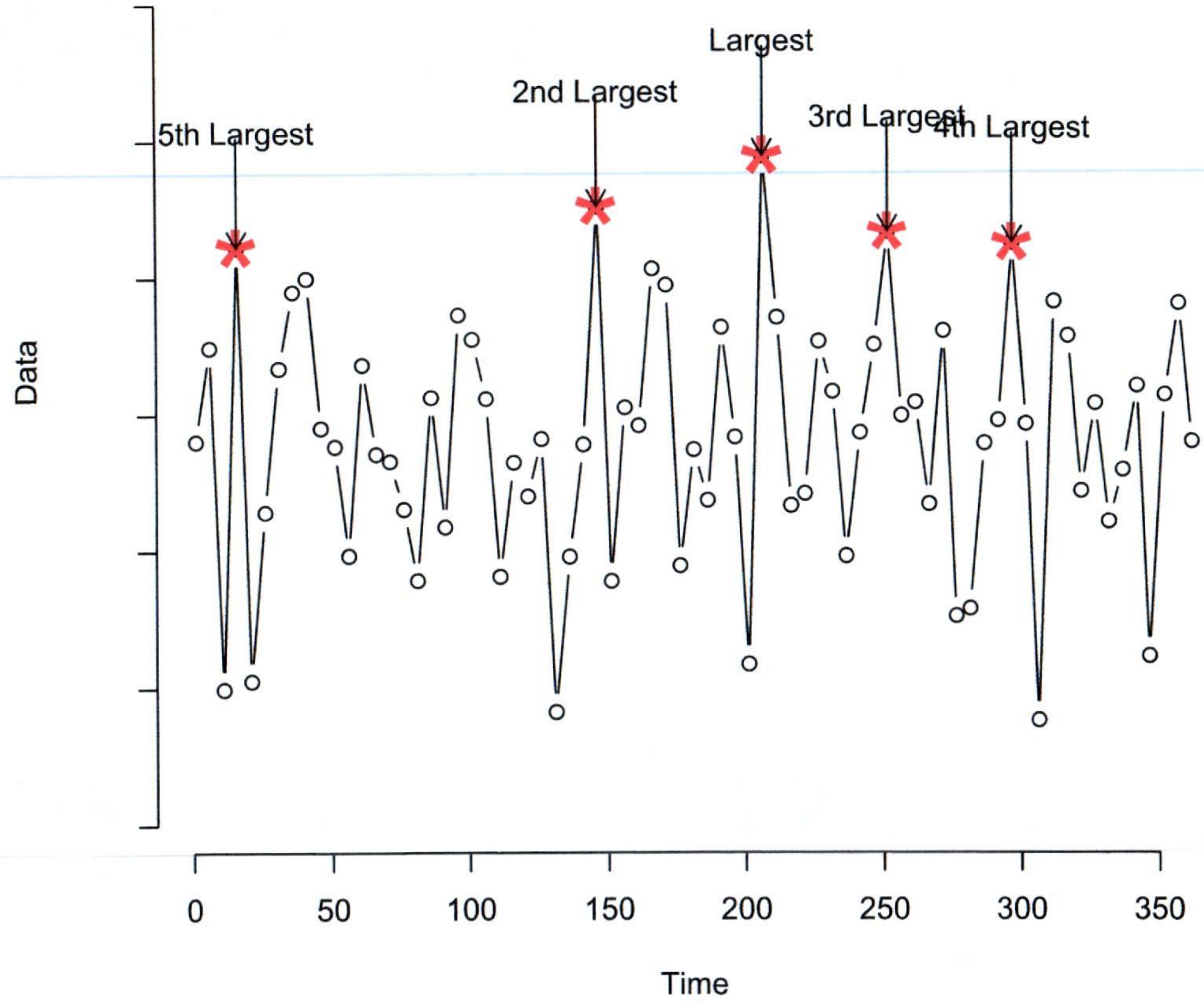
Definition 3

Extreme values = first few largest data values



Definition 3 of extreme values

Let

$M_n^{(1)}$ = Largest value

$M_n^{(2)}$ = 2^{nd} " "

$M_n^{(3)}$ = 3^{rd} " "

$\vdots$

$M_n^{(r)}$ = rth " "

What is the distribution of $(M_n^{(1)}, \ldots, M_n^{(r)})$?

A result due to Weissman (1978) is

$$P\left[\frac{M_n^{(1)} - b_n}{a_n} < x_1, \ldots, \frac{M_n^{(r)} - b_n}{a_n} < x_r\right]$$

$$\rightarrow \sum_{s_1=0}^{1} \sum_{s_2=0}^{2-s_1} \cdots \sum_{s_{r-1}=0}^{r-1-s_1-\cdots-s_{r-2}} \frac{(\gamma_2-\gamma_1)^{s_1}}{s_1!} \cdots \frac{(\gamma_r-\gamma_{r-1})^{s_{r-1}}}{s_{r-1}!} e^{-\gamma_r} \quad \cdots (*)$$

as $n \rightarrow \infty$, where

$\gamma_i = (1+\xi x_i)^{-\frac{1}{\xi}}$

a_n = same as in ETT

b_n = same as in ETT

It can be shown from (*) that the joint PDF of $(M_n^{(1)}, \dots, M_n^{(r)})$ is

$$\sigma^{-r} \exp\left\{ -\left(1+\xi\,\frac{x_r-\mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{r} \log\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right) \right\}$$

for $x_1 \geq x_2 \geq \dots \geq x_r$ and

$$1+\xi\,\frac{x_i-\mu}{\sigma} > 0, \quad i=1,2,\dots,r$$

Estimation under definition 3

Suppose

$(x_{1,1}, x_{1,2}, \dots, x_{1,r})$ for 1^{st} year

$(x_{2,1}, x_{2,2}, \dots, x_{2,r})$ for 2^{nd} year

$\vdots$

$(x_{T,1}, x_{T,2}, \dots, x_{T,r})$ for T^{th} year.

The data are the r largest observations for T years. The likelihood function is

$$L(\mu, \sigma, \xi) = \prod_{i=1}^{T} \left[\sigma^{-r} \exp\left\{ - \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(\frac{1}{\xi} + 1\right) \sum_{j=1}^{r} \log\left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right) \right\} \right]$$

$$= \sigma^{-nr} \exp\left\{ - \sum_{i=1}^{T} \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{T} \sum_{j=1}^{r} \log\left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right) \right\}$$

The log-likelihood function is

$$\log L(\mu, \sigma, \xi) = -Tr \log \sigma - \sum_{i=1}^{T} \left(1 + \xi \frac{x_{i,r} - \mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{T} \sum_{j=1}^{r} \log \left(1 + \xi \frac{x_{i,j} - \mu}{\sigma}\right).$$

The MLEs of μ, σ and ξ are the simultaneous solutions

$$\frac{\partial \log L}{\partial \mu} = 0,$$

$$\frac{\partial \log L}{\partial \sigma} = 0,$$

$$\frac{\partial \log L}{\partial \xi} = 0.$$

MLE equations for the r largest distribution

The MLEs of μ, σ and ξ are the simultaneous solutions of

$$\begin{aligned}\frac{\partial \log L}{\partial \mu} &= -\frac{1}{\sigma}\sum_{i=1}^{T}\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad -\frac{1+\xi}{\sigma}\sum_{i=1}^{T}\sum_{j=1}^{r}\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0, \qquad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} &= -\frac{Tr}{\sigma}-\frac{1}{\sigma^2}\sum_{i=1}^{T}\left(x_{i,r}-\mu\right)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad +\frac{1+\xi}{\sigma^2}\sum_{i=1}^{T}\sum_{j=1}^{r}\left(x_{i,j}-\mu\right)\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0, \qquad (2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \xi} &= -\frac{1}{\xi^2}\sum_{i=1}^{T}\log\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}} \\ &\quad +\frac{1}{\xi\sigma}\sum_{i=1}^{T}\left(x_{i,r}-\mu\right)\left(1+\xi\frac{x_{i,r}-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \\ &\quad +\frac{1}{\xi^2}\sum_{i=1}^{T}\sum_{j=1}^{r}\log\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right) \\ &\quad -\frac{1+\xi}{\xi\sigma}\sum_{i=1}^{T}\sum_{j=1}^{r}\left(x_{i,j}-\mu\right)\left(1+\xi\frac{x_{i,j}-\mu}{\sigma}\right)^{-1} \\ &= 0. \qquad (3)\end{aligned}$$

The MLEs of μ, σ and ξ are the simultaneous solutions (1), (2) and (3).