How to know which of the 3 limits is attained (if there is a limit)?

- type I (Gumbel type) will be attained if there exists $\gamma(t) > 0$ such that

$$\lim_{t \to w(F)} \frac{1-F(t+x\gamma(t))}{1-F(t)} = e^{-x}$$

- type II (Fréchet type) will be attained if $w(F) = \infty$ and

$$\lim_{t \to \infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}, \quad \alpha > 0$$

- type III (Weibull type) will be attained if $w(F) < \infty$ and

$$\lim_{t \to 0} \frac{1-F(w(F)-tx)}{1-F(w(F)-t)} = x^{\alpha}, \quad \alpha > 0$$

Only one of these three conditions will ever be satisfied.

## Definition of $w(F)$

$w(F) =$ " upper end point " of the support of $F$.

Formally,

$$w(F) = \sup\{x : F(x) < 1\}$$

An easy way to find $w(F)$ is to set $F(x) = 1$ and then solve for $x$.

**Example 1** $F(x) = 1 - e^{-x} = 1$

$$\Rightarrow e^{-x} = 0$$

$$\Rightarrow -x = -\infty$$

$$\Rightarrow x = +\infty$$

$$\Rightarrow w(F) = +\infty$$

**Example 2**

$$F(x) = 1 - \frac{1}{x} = 1$$

$$\Rightarrow \frac{1}{x} = 0$$

$$\Rightarrow x = +\infty$$

$$\Rightarrow w(F) = +\infty$$

**Example 3** $F(x) = x, \quad 0 < x < 1$

$$x = 1$$

$$\Rightarrow w(F) = 1$$

## Examples on checking conditions I-III

**Example 1** Suppose $F(x) = 1 - e^{-x}$

We know $\omega(F) = +\infty$.

$$\text{I}: \quad \lim_{t \to \omega(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - [\cancel{1} - e^{-t - x\gamma(t)}]}{\cancel{1} - [\cancel{1} - e^{-t}]}$$

$$= \lim_{t \to \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if} \quad \gamma(t) \equiv 1 \quad \forall t$$

Hence, the Gumbel type is attained.

That is, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$[F(a_n x + b_n)]^n \longrightarrow e^{-e^{-x}}$$

as $n \longrightarrow \infty$.

Example 2 Suppose $F(x) = 1 - \frac{1}{x}$.

$$w(F) = +\infty.$$

$$\text{I} : \lim_{t \to \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{t + x\gamma(t)}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{t}\right)}$$

$$= \lim_{t \to \infty} \frac{t}{t + x\gamma(t)}$$

$$\neq e^{-x}$$

$\Rightarrow$ condition I fails to hold.

$\mathrm{II}$ : $\displaystyle \lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{tx}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{t}\right)}$$

$$= \lim_{t \to \infty} \frac{\frac{1}{tx}}{\frac{1}{t}}$$

$$= \frac{1}{x}$$

$$= x^{-\alpha} \quad \text{when} \quad \alpha = 1$$

Hence, condition $\underline{\mathrm{II}}$ holds with $\alpha = 1$. That is, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$[F_n(b_n + a_n x)]^n \longrightarrow \begin{cases} 0 & \text{if } x < 0 \\ e^{-x^{-1}} & \text{if } x \geq 0 \end{cases}$$

as $n \longrightarrow \infty$.

Example 3 Suppose $F(x) = x, 0<x<1$, the CDF of $Uni[0,1]$

$$w(F) = 1$$

I : $\lim\limits_{t \to 1} \dfrac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$$= \lim_{t \to 1} \frac{1 - (t + x\gamma(t))}{1 - t}$$

$$= \lim_{t \to 1} 1 - \frac{x\gamma(t)}{1-t}$$

$$\neq e^{-x}$$

$\Rightarrow$ condition I fails to hold

II : $w(F) = 1 < \infty$

$\Rightarrow$ condition II fails to hold.

III : $\lim\limits_{t \to 0} \dfrac{1 - F(1 - tx)}{1 - F(1-t)}$

$$= \lim_{t \to 0} \frac{\cancel{1} - (\cancel{1} - tx)}{\cancel{1} - (\cancel{1} - t)}$$

$$= \lim_{t \to 0} \frac{tx}{t}$$

$= x = x^{\alpha}$ with $\alpha = 1$.

$\Rightarrow$ condition III holds with $\alpha = 1$.

Hence, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$[F(a_n x + b_n)]^n \rightarrow \begin{cases} e^x & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$$

as $n \rightarrow \infty$.