Case 5 of Portfolio Theory

[$X_1, \ldots, X_n$ are dependent RVs & n is a fixed]

a) The CDF of S' is

$$F_{S'}(s) = P(S' \leqslant s)$$

$$= \int_{\sum_{i=1}^{n} x_i \leqslant s} \underbrace{\boxed{f_{X_1, \ldots, X_n}(x_1, \ldots, x_n)}}_{\text{Joint PDF of } (X_1, \ldots, X_n)} dx_n \ldots dx_1$$

The PDF of S' is

$$f_{S'}(s) = \frac{d}{ds} F_{S'}(s).$$

Ex 12 Suppose $n=2$. The CDF of S is

$$F_S(s) = \int_{x_1+x_2 \leq s} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{s-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1 .$$

The PDF of S is

$$f_S(s) = \frac{d}{ds} F_S(s)$$

$$= \frac{d}{ds} \int_{-\infty}^{\infty} \int_{-\infty}^{s-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

$$= \int_{-\infty}^{\infty} \frac{d}{ds} \int_{-\infty}^{s-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

$$= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, s-x_1)\,dx_1 .$$

b) The CDF of U is

$$F_U(u) = P(U \leq u)$$

$$= 1 - P(U > u)$$

$$= 1 - P(\min(X_1, \dots, X_n) > u)$$

$$= 1 - P(X_1 > u, \dots, X_n > u)$$

$$= 1 - \boxed{\overline{F}_{X_1, \dots, X_n}}(u, \dots, u)$$

Joint survival function of $(X_1, \dots, X_n)$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= -\frac{d}{du} \overline{F}_{X_1, \dots, X_n}(u, \dots, u).$$

c) The cdf of V is

$$F_V(v) = P(V \leq v)$$

$$= P(\max(X_1, \ldots, X_n) \leq v)$$

$$= P(X_1 \leq v, \ldots, X_n \leq v)$$

$$= \boxed{F_{X_1, \ldots, X_n}}(v, \ldots, v)$$

Joint CDF of $(X_1, \ldots, X_n)$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \frac{d}{dv} F_{X_1, \ldots, X_n}(v, \ldots, v).$$

Ex 13 Suppose $n=2$ and (X_1, X_2) has the joint survival function

$$\bar{F}_{X_1, X_2}(x_1, x_2) = e^{-x_1 - x_2 - \theta x_1 x_2}$$

for $x_1 > 0$ and $x_2 > 0$. Find the following

(i) CDF of S'

(ii) PDF of S'

(iii) CDF of U

(iv) PDF of U

(v) CDF of V

(vi) PDF of V

(i) The CDF of S' is

$$F_{S'}(s) = \int_0^\infty \int_0^{s-x_1} f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$$

by Ex 12

By example 11,

$$f_{X_1, X_2}(x_1, x_2) = \left[(1+\theta x_1)(1+\theta x_2) - \theta\right] \cdot e^{-x_1 - x_2 - \theta x_1 x_2}.$$

$$\Rightarrow F_{S'}(s) = \int_0^\infty \int_0^{s-x_1} \left[(1+\theta x_1)(1+\theta x_2) - \theta\right] \cdot e^{-x_1 - x_2 - \theta x_1 x_2}\, dx_2\, dx_1$$

= please work this out

(ii) By example 12,

$$f_S(s) = \int_0^\infty f_{X_1, X_2}(x_1, s-x_1)\, dx_1$$

$$= \int_0^\infty \left[(1+\theta x_1)(1+\theta(s-x_1)) - \theta \right] \cdot e^{-x_1-(s-x_1)-\theta x_1(s-x_1)}\, dx_1$$

$$= e^{-s} \int_0^\infty \left[(1+\theta x_1)(1+\theta(s-x_1)) - \theta \right] \cdot e^{-\theta x_1(s-x_1)}\, dx_1$$

= please work this out

(iii) The CDF of U is

$$F_U(u) = 1 - \bar{F}_{X_1, X_2}(u, u)$$

$$= 1 - e^{-u-u-\theta u^2}$$

$$= 1 - e^{-2u-\theta u^2}$$

(iv) The PDF of U is

$$f_U(u) = \frac{d}{du}\left[1 - e^{-2u-\theta u^2}\right]$$

$$= 2(1+\theta u)\, e^{-2u-\theta u^2}.$$

(v) The CDF of V is

$$F_V(v) = F_{X_1, X_2}(v, v)$$

$$= 1 - \bar{F}_{X_1}(v) - \bar{F}_{X_2}(v) + \bar{F}_{X_1, X_2}(v, v)$$

$$= 1 - e^{-v} - e^{-v} + e^{-v-v-\theta v^2}$$

$$= 1 - 2e^{-v} + e^{-2v-\theta v^2}.$$

(vi) The PDF of V is

$$f_V(v) = \frac{d}{dv}\left[1 - 2e^{-v} + e^{-2v-\theta v^2}\right]$$

$$= 2e^{-v} - 2(1+\theta v)\, e^{-2v-\theta v^2}$$