Case 3 of Portfolio Theory

[$X_1, \ldots, X_n$ are INID & n is fixed]

a) The CDF of S is

$$F_S(s) = P(S \leq s)$$

independence

$$= \int_{\sum_{i=1}^{n} x_i \leq s} f_{X_1}(x_1) \cdots f_{X_n}(x_n) \, dx_n \cdots dx_1$$

$$= \int_{\sum_{i=1}^{n} x_i \leq s} \prod_{i=1}^{n} f_{X_i}(x_i) \, dx_n \cdots dx_1 .$$

The PDF of S is

$$f_S(s) = \frac{d}{ds} F_S(s).$$

<u>Ex 6</u> Suppose $n = 2$. The CDF of $\mathcal{S}$ is

$$F_{\mathcal{S}}(s) = P(X_1 + X_2 \le s)$$

$$\overset{\text{independence}}{=} \int_{x_1 + x_2 \le s} f_{X_1}(x_1)\, f_{X_2}(x_2)\, dx_2\, dx_1$$

$$= \int_{-\infty}^{\infty} f_{X_1}(x_1) \underbrace{\int_{-\infty}^{s - x_1} f_{X_2}(x_2)\, dx_2}_{= F_{X_2}(s - x_1)}\, dx_1$$

$$= \int_{-\infty}^{\infty} f_{X_1}(x_1)\, F_{X_2}(s - x_1)\, dx_1 .$$

The PDF of $\mathcal{S}$ is

$$f_{\mathcal{S}}(s) = \frac{d}{ds} F_{\mathcal{S}}(s)$$

$$= \int_{-\infty}^{\infty} f_{X_1}(x_1)\, f_{X_2}(s - x_1)\, dx_1 .$$

b) The CDF of U is

$$F_U(u) = P(U \leq u)$$

$$= 1 - P(U > u)$$

$$= 1 - P(\min(X_1, \ldots, X_n) > u)$$

$$= 1 - P(X_1 > u, \ldots, X_n > u)$$

independence

$$= 1 - P(X_1 > u) \cdots P(X_n > u)$$

$$= 1 - [1 - P(X_1 \leq u)] \cdots [1 - P(X_n \leq u)]$$

$$= 1 - [1 - F_{X_1}(u)] \cdots [1 - F_{X_n}(u)]$$

$$= 1 - \prod_{i=1}^{n} [1 - F_{X_i}(u)].$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{i=1}^{n} f_{X_i}(u) \prod_{\substack{j=1 \\ j \neq i}}^{n} [1 - F_{X_j}(u)]$$

Product rule

c) The CDF of V is

$$F_V(v) = P(V \leq v)$$

$$= P\left[\max(X_1, \ldots, X_n) \leq v\right]$$

$$= P\left[X_1 \leq v, \ldots, X_n \leq v\right]$$

independence

$$= P[X_1 \leq v] \cdots P[X_n \leq v]$$

$$= F_{X_1}(v) \cdots F_{X_n}(v)$$

$$= \prod_{i=1}^{n} F_{X_i}(v)$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \sum_{i=1}^{n} f_{X_i}(v) \prod_{\substack{j=1 \\ j \neq i}}^{n} F_{X_j}(v)$$

Product Rule

Product Rules for Differentiation

1) $$\frac{d}{dx}\left[g_1(x)\, g_2(x)\right] = g_1(x)\, g_2'(x) + g_1'(x)\, g_2(x)$$

2) $$\frac{d}{dx}\left[g_1(x)\, g_2(x)\, g_3(x)\right]$$
$$= g_1(x)\, g_2(x)\, g_3'(x)$$
$$+ g_1(x)\, g_2'(x)\, g_3(x)$$
$$+ g_1'(x)\, g_2(x)\, g_3(x).$$

3) $$\frac{d}{dx}\left[\prod_{i=1}^{n} g_i(x)\right]$$
$$= \sum_{i=1}^{n} g_i'(x) \prod_{\substack{j=1 \\ j \neq i}}^{n} g_j(x)$$

Ex 7 Suppose $n = 2$. The CDF of U is

$$F_U(u) = 1 - [1 - F_{X_1}(u)][1 - F_{X_2}(u)]$$

$$= F_{X_1}(u) + F_{X_2}(u) - F_{X_1}(u) F_{X_2}(u).$$

The PDF of U is

$$f_U(u) = f_{X_1}(u) + f_{X_2}(u) - f_{X_1}(u) F_{X_2}(u) - F_{X_1}(u) f_{X_2}(u)$$

Ex 8 Suppose $n = 2$. The CDF of V is

$$F_V(v) = F_{X_1}(v)\, F_{X_2}(v).$$

The PDF of V is

$$f_V(v) = f_{X_1}(v)\, F_{X_2}(v) + F_{X_1}(v)\, f_{X_2}(v).$$

Ex 9 Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ are independent RVs. Find the distributions of S, U & V.

a) $S = X_1 + \cdots + X_n$

$$\sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

The CDF of S is

$$F_S(s) = \Phi\left(\frac{s - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}\right),$$

where $\Phi(\cdot)$ denotes the CDF of $N(0,1)$.

The PDF of S is

$$f_S(s) = \frac{1}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \phi\left(\frac{s - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}\right),$$

where $\phi(\cdot)$ denotes the PDF of $N(0,1)$.

b) The CDF of U is

$$F_U(u) = 1 - \prod_{i=1}^{n} \left[1 - \Phi\left(\frac{u - \mu_i}{\sigma_i} \right) \right]$$

$$\boxed{1 - \Phi(z) = \Phi(-z)}$$

$$= 1 - \prod_{i=1}^{n} \Phi\left(\frac{\mu_i - u}{\sigma_i} \right)$$

c) The CDF of V is

$$F_V(v) = \prod_{i=1}^{n} \Phi\left(\frac{v - \mu_i}{\sigma_i} \right).$$