Case 2 of Portfolio Theory

($X_1, \dots, X_n$ are IID & n is a RV independent of $X_1, \dots, X_n$)

a) The CDF of S' is

$$F_{S'}(s) = P(S' \leq s)$$

$$= \sum_{n=1}^{\infty} P(S' \leq s \mid N = n)\, P(N = n)$$

Total Prob Rule

Case 1

$$= \sum_{n=1}^{\infty} \left[\int_{\sum_{i=1}^{n} x_i \leq s} \prod_{i=1}^{n} f_X(x_i)\, dx_n \cdots dx_2\, dx_1 \right] \cdot P(N = n)$$

The PDF of S' is

$$f_{S'}(s) = \frac{d}{ds} F_{S'}(s).$$

b) The CDF of U is

$$F_U(u) = P(U \leq u)$$

$$= \sum_{n=1}^{\infty} P(U \leq u \mid N=n)\, P(N=n)$$

Total Prob Rule

Case 1

$$= \sum_{n=1}^{\infty} \left\{1 - [1 - F_X(u)]^n\right\} P(N=n)$$

$$= \sum_{n=1}^{\infty} P(N=n) - \sum_{n=1}^{\infty} [1 - F_X(u)]^n P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} [1 - F_X(u)]^n P(N=n)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{n=1}^{\infty} n [1 - F_X(u)]^{n-1} f_X(u)\, P(N=n)$$

c) The CDF of V is

$$F_V(v) = P(V \leq v)$$

$$\overset{\boxed{\text{Total}}\ \boxed{\text{Prob}}\ \boxed{\text{Rule}}}{=} \sum_{n=1}^{\infty} P(V \leq v \mid N=n)\, P(N=n)$$

$$\overset{\boxed{\text{Case 1}}}{=} \sum_{n=1}^{\infty} [F_X(v)]^n \, P(N=n).$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \sum_{n=1}^{\infty} n \, [F_X(v)]^{n-1} f_X(v) \, P(N=n)$$

Ex 4 Suppose $X_1, \ldots, X_N$ are IID with $N \sim \text{Geom}(a)$. Find the distributions of U & V.

The CDF of U is

$$F_U(u) = 1 - \sum_{n=1}^{\infty} [1 - F_X(u)]^n \, a(1-a)^{n-1}$$

$$= 1 - a[1 - F_X(u)] \sum_{n=1}^{\infty} \{[1-F_X(u)](1-a)\}^{n-1}$$

$\boxed{\text{Set } m = n-1}$

$$= 1 - a[1 - F_X(u)] \sum_{m=0}^{\infty} \{[1-F_X(u)](1-a)\}^{m}$$

$\boxed{\sum_{m=0}^{\infty} z^m = \frac{1}{1-z}}$

$$= 1 - \frac{a[1 - F_X(u)]}{1 - (1-a)[1 - F_X(u)]}$$

The CDF of V is

$$F_V(v) = \sum_{n=1}^{\infty} [F_X(v)]^n a(1-a)^{n-1}$$

$$\boxed{\text{Set} \quad m = n-1}$$

$$= a F_X(v) \sum_{m=0}^{\infty} [F_X(v)(1-a)]^m$$

$$\boxed{\sum_{m=0}^{\infty} z^m = \frac{1}{1-z}}$$

$$= \frac{a F_X(v)}{1-(1-a)F_X(v)}.$$

Ex 5 Suppose $X_1, \dots, X_N$ are IID with

$$P(N=n) = \frac{e^{-a} a^n}{(1-e^{-a})\, n!}, \quad n = 1, 2, \dots$$

Truncated Poisson

Find the distributions of U & V.

The CDF of U is

$$F_U(u) = 1 - \sum_{n=1}^{\infty} [1 - F_X(u)]^n \frac{e^{-a} a^n}{(1-e^{-a})\, n!}$$

$$= 1 - \frac{e^{-a}}{1-e^{-a}} \sum_{n=1}^{\infty} \frac{\{[1 - F_X(u)]\, a\}^n}{n!}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{z^n}{n!} = e^z - 1}$$

$$= 1 - \frac{1}{e^a - 1} \left[e^{[1 - F_X(u)]\, a} - 1 \right]$$

The CDF of V is

$$F_V(v) = \sum_{n=1}^{\infty} [F_X(v)]^n \frac{e^{-a} a^n}{(1-e^{-a})\, n!}$$

$$= \frac{e^{-a}}{1-e^{-a}} \sum_{n=1}^{\infty} \frac{\{F_X(v)\, a\}^n}{n!}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{z^n}{n!} = e^z - 1}$$

$$= \frac{1}{e^a - 1} \left[e^{F_X(v)\, a} - 1 \right].$$