Case 1 of Portfolio Theory

($X_1, \ldots, X_n$ are IID & n is fixed)

a) The CDF of S' is

$$F_{S'}(s) = P(S' \leq s)$$

$X_1, \ldots, X_n$ are indep RVs

$$= \int_{\sum_{i=1}^{n} x_i \leq s} f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(x_n)\, dx_n \cdots dx_2\, dx_1$$

$X_1, \ldots, X_n$ are identical RVs

$$= \int_{\sum_{i=1}^{n} x_i \leq s} \left[\prod_{i=1}^{n} f_X(x_i) \right] dx_n \cdots dx_2\, dx_1$$

The PDF of S' is

$$f_{S'}(s) = \frac{d}{ds} F_{S'}(s).$$

Ex 1   Suppose $n = 2$. The CDF of $\mathcal{S}$ is

$$F_{\mathcal{S}}(s) = P(X_1 + X_2 \leq s)$$

$$= \int_{x_1 + x_2 \leq s} f_{X_1}(x_1)\, f_{X_2}(x_2)\, dx_2\, dx_1$$

$$= \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{s - x_1} f_{X_1}(x_1)\, f_{X_2}(x_2)\, dx_2}_{= f_{X_1}(x_1) F_{X_2}(s - x_1)}\, dx_1$$

$$= \int_{-\infty}^{\infty} f_{X_1}(x_1)\, F_{X_2}(s - x_1)\, dx_1$$

$\boxed{X_1 \,\&\, X_2 \text{ are identical RVs}}$

$$= \int_{-\infty}^{\infty} f_X(x_1)\, F_X(s - x_1)\, dx_1$$

The PDF of $\mathcal{S}$ is

$$f_{\mathcal{S}}(s) = \frac{d}{ds} F_{\mathcal{S}}(s)$$

$$= \frac{d}{ds} \int_{-\infty}^{\infty} f_X(x_1)\, F_X(s - x_1)\, dx_1$$

$$= \int_{-\infty}^{\infty} f_X(x_1)\, f_X(s - x_1)\, dx_1$$

b) The CDF of U is

$$
\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= P(\min(X_1, \ldots, X_n) \leq u) \\
&= 1 - P(\min(X_1, \ldots, X_n) > u) \\
&= 1 - P(X_1 > u, \ldots, X_n > u) \\
&\overset{\text{independence}}{=} 1 - P(X_1 > u) \cdots P(X_n > u) \\
&= 1 - [1 - P(X_1 \leq u)] \cdots [1 - P(X_n \leq u)] \\
&= 1 - [1 - F_{X_1}(u)] \cdots [1 - F_{X_n}(u)] \\
&\overset{\text{identical}}{=} 1 - [1 - F_X(u)] \cdots [1 - F_X(u)] \\
&= 1 - [1 - F_X(u)]^n
\end{aligned}
$$

The PDF of U is

$$
\begin{aligned}
f_U(u) &= \frac{d}{du} F_U(u) \\
&= n\,[1 - F_X(u)]^{n-1} f_X(u).
\end{aligned}
$$

c) The CDF of V is

$$F_V(v) = P(V \leq v)$$

$$= P(\max(X_1, \ldots, X_n) \leq v)$$

$$= P(X_1 \leq v, \ldots, X_n \leq v)$$

independence

$$= P(X_1 \leq v) \cdots P(X_n \leq v)$$

$$= F_{X_1}(v) \cdots F_{X_n}(v)$$

identical

$$= F_X(v) \cdots F_X(v)$$

$$= [F_X(v)]^n.$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= n[F_X(v)]^{n-1} f_X(v)$$

Ex 2

Suppose $X_1, \ldots, X_n$ are IID $N(\mu, \sigma^2)$.
Find the distribution of S, U and V.

a) $\Rightarrow S = X_1 + \cdots + X_n \sim N(n\mu, n\sigma^2)$

The CDF of S is

$$F_S(s) = \Phi\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right)$$

where $\Phi(\cdot)$ denotes the CDF of $N(0, 1)$.

The PDF of S is

$$f_S(s) = \frac{1}{\sqrt{n}\,\sigma}\,\phi\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right)$$

$$= \frac{1}{\sqrt{n}\,\sigma}\,\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{s - n\mu}{\sqrt{n}\,\sigma}\right)^2\right]$$

where $\phi(\cdot)$ denotes the PDF of $N(0, 1)$.

b) The CDF of U is

$$F_U(u) = 1 - \left[1 - \Phi\left(\frac{u-\mu}{\sigma}\right)\right]^n.$$

The PDF of U is

$$f_U(u) = n\left[1 - \Phi\left(\frac{u-\mu}{\sigma}\right)\right]^{n-1} \cdot \frac{1}{\sigma}\phi\left(\frac{u-\mu}{\sigma}\right)$$

c) The CDF of V is

$$F_V(v) = \left[\Phi\left(\frac{v-\mu}{\sigma}\right)\right]^n.$$

The PDF of V is

$$f_V(v) = n\left[\Phi\left(\frac{v-\mu}{\sigma}\right)\right]^{n-1}\frac{1}{\sigma}\phi\left(\frac{v-\mu}{\sigma}\right)$$

Ex 3

Suppose $X_1, \dots, X_n$ are IID $Exp(a)$. Find the distributions of S, U & V.

a) $S = X_1 + \dots + X_n \sim Gamma(n, a)$.

The CDF of S is

$$F_S(s) = \frac{\gamma(n, as)}{\Gamma(n)}$$

where

$$\gamma(n, z) = \int_0^z t^{n-1} e^{-t} \, dt$$

$\boxed{\text{incomplete gamma function}}$

The PDF of S is

$$f_S(s) = \frac{a^n s^{n-1} e^{-as}}{\Gamma(n)}.$$

b) The CDF of U is

$$F_U(u) = 1 - \left[1 - \left(1 - e^{-au}\right)\right]^n$$

$$= 1 - e^{-nau}$$

The PDF of U is

$$f_U(u) = na\,e^{-nau}$$

c) The CDF of V is

$$F_V(v) = \left[1 - e^{-av}\right]^n$$

The PDF of V is

$$f_V(v) = n\left[1 - e^{-av}\right]^{n-1} a\,e^{-av}.$$