Alternative conditions to check which of the 3 limits is attained

Let
$$f(t) = \frac{dF(t)}{dt},$$
$$a(t) = F^{-1}\left(1 - \frac{1}{t}\right),$$
$$b(t) = t\, f(a(t)).$$

I : $\displaystyle\lim_{t \to \infty} \frac{b(tx)}{b(t)} = 1$

II : $w(F) = \infty$ and $\displaystyle\lim_{t \to \infty} a(t)\, b(t) = \alpha > 0$

III : $w(F) < \infty$ and $\displaystyle\lim_{t \to \infty} \{w(F) - a(t)\}\, b(t) = \alpha > 0$

These conditions are equivalent to the previous conditions.
Once again, only one of the conditions will be ever satisfied.

Example 1 $\quad F(x) = 1 - e^{-x}$

$$f(x) = \frac{dF(x)}{dx} = e^{-x}$$

Set $F(x) = 1 - \frac{1}{t}$

$$\Rightarrow 1 - e^{-x} = 1 - \frac{1}{t}$$

$$\Rightarrow e^{-x} = \frac{1}{t}$$

$$\Rightarrow -x = -\log t$$

$$\Rightarrow x = \log t$$

$$\Rightarrow a(t) = \log t$$

$$\begin{aligned} b(t) &= t \cdot f(a(t)) \\ &= t \cdot e^{-a(t)} \\ &= t \cdot e^{-\log t} \\ &= t \cdot \frac{1}{t} \\ &= 1 \end{aligned}$$

$$\mathrm{I}: \lim_{t \to \infty} \frac{b(tx)}{b(t)} = \lim_{t \to \infty} \frac{1}{1} = 1$$

Hence, condition I holds

Example 2 $\quad F(x) = 1 - \frac{1}{x}$.

$$w(F) = \infty$$

$$f(x) = \frac{dF(x)}{dx} = \frac{1}{x^2}$$

$$F(x) = 1 - \frac{1}{t}$$

$$\Rightarrow \quad 1 - \frac{1}{x} = 1 - \frac{1}{t}$$

$$\Rightarrow \quad x = t$$

$$\Rightarrow \quad a(t) = t$$

$$b(t) = t \cdot f(a(t))$$

$$= t \cdot \frac{1}{(a(t))^2}$$

$$= t \cdot \frac{1}{t^2}$$

$$= \frac{1}{t}.$$

I : $\lim_{t \to \infty} \frac{b(tx)}{b(t)} = \lim_{t \to \infty} \frac{\frac{1}{tx}}{\frac{1}{t}} = \frac{1}{x} \neq 1$

$\Rightarrow$ condition I fails to hold.

II : $w(F) = \infty$ ✓

$\lim_{t \to \infty} a(t)\, b(t) = \lim_{t \to \infty} t \cdot \frac{1}{t} = 1 > 0$

Hence, condition II holds with $\alpha = 1$

Example 3 $\quad F(x) = x, \quad 0 < x < 1$

$$w(F) = 1$$

$$f(x) = \frac{dF(x)}{dx} = 1$$

$$F(x) = 1 - \frac{1}{t}$$

$$\Rightarrow \quad x = 1 - \frac{1}{t}$$

$$\Rightarrow \quad a(t) = 1 - \frac{1}{t}$$

$$b(t) = t \cdot f(a(t))$$

$$= t \cdot 1$$

$$= t$$

I : $\lim_{t \to \infty} \frac{b(tx)}{b(t)} = \lim_{t \to \infty} \frac{tx}{t} = x \neq 1$

$\Rightarrow$ condition I fails to hold

II : $w(F) = 1 \neq \infty$

$\Rightarrow$ condition II fails to hold

III : $w(F) = 1 < \infty \quad \checkmark$

$$\lim_{t \to \infty} \{1 - a(t)\} \, b(t)$$

$$= \lim_{t \to \infty} \{1 - (1 - \tfrac{1}{t})\} \, t$$

$$= 1$$

Hence, condition III holds with $\alpha = 1$.